JOURNAL OF COMBINATORIAL THEORY, Series B 24, 94–123 (1978)

Orientability of Matroids

ROBERT G. BLAND

Center for Operations Research and Econometrics
and
The State University of New York, Binghamton, New York 13901

AND

MICHEL LAS VERGNAS

CNRS, Paris, France

Communicated by the Editors

Received June 11, 1974

In this paper we define oriented matroids and develop their fundamental properties, which lead to generalizations of known results concerning directed graphs, convex polytopes, and linear programming. Duals and minors of oriented matroids are defined. It is shown that every coordinatization (representation) of a matroid over an ordered field induces an orientation of the matroid. Examples of matroids that are orientable but not coordinatizable and of matroids that are not orientable are presented. We show that a binary matroid is orientable if and only if it is unimodular (regular), and that every unimodular matroid has an orientation that is induced by a coordinatization and is unique in a certain straightforward sense.

1. INTRODUCTION

Let F be a field, let E be a finite set, and denote by F^E the vector space of mappings from E to F. The *support* of $\alpha \in F^E$ is defined to be the set $\underline{S}(\alpha) = \{e \in E: \alpha(e) \neq 0\}$.

Let $\mathscr{R}$ be a vector subspace of F^E. A nonzero vector $\alpha \in \mathscr{R}$ is an *elementary* vector of $\mathscr{R}$ if $\underline{S}(\alpha)$ is minimal (with respect to inclusion) among the supports of all nonzero vectors in $\mathscr{R}$. The set $\mathscr{C}$ of supports of elementary vectors of $\mathscr{R}$ has the following properties:

(C1) $C \in \mathscr{C}$ implies $C \neq \varnothing$, and

$$C_1, C_2 \in \mathscr{C} \text{ and } C_1 \subseteq C_2 \text{ imply } C_1 = C_2;$$

0095-8956/78/0241-0094$02.00/0

(C2) for all $C_1, C_2 \in \mathscr{C}$ and $x \in C_1 \cap C_2$, $y \in C_1 \backslash C_2$, there exists $C_3 \in \mathscr{C}$ such that $y \in C_3 \subseteq (C_1 \cup C_2) \backslash x$.

Whitney [15] used the properties (C1) and (C2) to abstract linear dependence, calling a set E together with *a* set $\mathscr{C}$ of subsets of E satisfying (C1) and (C2) a *matroid.* (The term *combinatorial pregeometry* is also used to describe such systems.) Not all matroids arise as above from vector spaces, yet matroids retain much of the fundamental structure of vector spaces. For example, the notions of rank, bases, flats, hyperplanes, and orthogonals generalize in the context of matroids. However, matroids do not capture certain sign properties of vector spaces over ordered fields. For example, let $G = (V, E)$ be a 2-connected graph, let A be the $(0, \pm 1)$-vertex-edge incidence matrix of an orientation of G and let $\mathscr{R}$ be the null space of A in $\mathbb{R}^E$. Then the orientation of G is lost in passing from the elementary vectors of $\mathscr{R}$ to their supports, but is deducible (up to reversing all edges) from the *signed supports* $(S^+(\alpha), S^-(\alpha))$ of the elementary $\alpha \in \mathscr{R}$, which distinguish the subsets $S^+(\alpha) = \{e \in E: \alpha(e) > 0\}$ and $S^-(\alpha) = \{e \in E : \alpha(e) < 0\}$ of $\underline{S}(\alpha)$.

In this paper we introduce and develop a theory of oriented matroids that generalizes the structure of signed supports of elementary vectors of a vector space over an ordered field. Oriented matroids thus provide a richer abstraction than matroids of vector spaces over ordered fields. In particular, one can generalize in the context of oriented matroids notions usually associated with oriented graphs, linear programming and convex polyhedra.

Camion [4], Fulkerson [8], and Rockafellar [13] previously investigated the combinatorial nature of a number of interesting theorems concerning vector spaces over ordered fields. Several of the theorems and proofs in [4, 8, 13] translate directly into the context of oriented matrois. In fact, Rockafellar in [13] suggested that one should be able to axiomatize a system of "signed" or "oriented" matroids that would abstract the combinatorial structure of signed supports of elementary vectors in ordered vector spaces. Minty's work on digraphoids [12], which gave the first notion of matroid orientations and partially motivated Camion, Fulkerson, and Rockafellar, was clearly too restrictive for this purpose. The broader notion of orientability presented here achieves the abstraction that Rockafellar foresaw.

In the next section we present five axiomatizations of oriented matroids and prove their equivalence. The subject of oriented matroid duality is naturally developed within the establishment of that equivalence. The remaining four sections concern: (3) examples and interpretations; (4) minors of oriented matroids; (5) systems whose minimal elements form an oriented matroid; and (6) binary oriented matroids (Minty's digraphoids [12]). It is assumed that the reader has some familiarity with matroid theory. Whitney's original paper on the subject [15], the paper by Tutte [14], and the book by Crapo and Rota [7] are appropriate references.

2. Oriented Matroids

We define a *signed set* X to be a set $\underline{X}$, called the set *underlying* X, and a mapping $sg_X(x) : \underline{X} \to \{-1, 1\}$, called the *signature of* X. Let X be a signed set. The sets $X^+ = \{x \in \underline{X} : sg_X(x) = 1\}$ and $X^- = \{x \in \underline{X} : sg_X(x) = -1\}$ describe X in a convenient way. The *opposite of* X, denoted $-X$, is the signed set having $(-X)^+ = X^-$ and $(-X)^- = X^+$; we write $Y = \pm X$ if either $Y = X$ or $Y = -X$. If $\underline{X}$ is a subset of some set E, then X will be called a *signed subset of* E, and if $\underline{X} = \varnothing$, then we write $X = \varnothing$.

Theorem 2.1. *Let E be a finite set and let $\mathcal{O}$ be a set of signed subsets of E such that*

(0) *for all $X \in \mathcal{O}$, $X \neq \varnothing$ and $-X \in \mathcal{O}$; and for all $X_1, X_2 \in \mathcal{O}$, $\underline{X}_2 \subseteq \underline{X}_1$ implies $X_1 = \pm X_2$.*

Then the following two properties are equivalent:

(I) *for all $X_1, X_2 \in \mathcal{O}$ such that $X_1 \neq -X_2$, and all $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$, there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x$;*

(II) *for all $X_1, X_2 \in \mathcal{O}$, $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ and $y \in (X_1^+\backslash X_2^-) \cup (X_1^-\backslash X_2^+)$, there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x$, $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x$, and $y \in \underline{X}_3$.*

Theorem 2.1 will be proved in the second part of this section.

We define an *oriented matroid* to be a structure $(E, \mathcal{O})$, as above, that satisfies (0) and (I).

For $\mathcal{O}$ a set of signed sets, let $\underline{\mathcal{O}} = \{\underline{X} : X \in \mathcal{O}\}$. If $M = (E, \mathcal{O})$ is an oriented matroid, then $\underline{M} = (E, \underline{\mathcal{O}})$ is a matroid, since (0) and (I) clearly imply Lehman's circuit axioms for $\underline{M}$ [11]. Note that (0) and (II) imply Whitney's circuit axioms [15]. If one relaxes (II) by requiring that $y \in \underline{X}_1\backslash\underline{X}_2$, rather than $y \in (X_1^+\backslash X_2^-) \cup (X_1^-\backslash X_2^+) \supseteq \underline{X}_1\backslash\underline{X}_2$, then the resulting property $(\mathrm{I}\,\frac{\mathrm{I}}{\mathrm{II}})$ is obviously stronger than (I) but weaker than (II), and is, by Theorem 2.1, equivalent to both, under condition (0). In the form $(\mathrm{I}\,\frac{\mathrm{I}}{\mathrm{II}})$, the elimination property for oriented matroids most closely resembles Whitney's circuit elimination axiom. In Section 5 we will see that when the condition

$$\underline{X}_2 \subseteq \underline{X}_1 \text{ implies } X_1 = \pm X_2$$

is dropped from (0), then (I) and (II) are no longer equivalent, while $(\mathrm{I}\,\frac{\mathrm{I}}{\mathrm{II}})$ and (II) are.

Let M be a matroid on E with circuits $\mathscr{C}$ and let $\mathcal{O}$ be a set of signed subsets of E. If $(E, \mathcal{O})$ is an oriented matroid and $\underline{\mathcal{O}} = \mathscr{C}$, then $\mathcal{O}$ is called an *orientation* of M and each $X \in \mathcal{O}$ is called a *(signed) circuit* of $(E, \mathcal{O})$. If there exists an orientation of M, then M is called *orientable*.

The key condition of the *signed elimination properties* (I) and (II) that relates to orientation is that

$$X_3^+ \subseteq X_1^+ \cup X_2^+ \qquad \text{and} \qquad X_3^- \subseteq X_1^- \cup X_2^-. \tag{2.1}$$

In the signed elimination properties, the underlying matroid structure and the structure pertaining specifically to orientation are intimately tied. By invoking matroid duality (orthogonality), one can define oriented matroids in such a way that properties pertaining solely to orientation are divorced in a natural way from those properties that stem only from the underlying matroid structure.

Let $M = (E, \mathscr{C})$ be a matroid. A set $\mathcal{O}$ of signed sets satisfying $\underline{\mathcal{O}} = \mathscr{C}$ and $\mathcal{O} = -\mathcal{O} = \{-X : X \in \mathcal{O}\}$ will be called a *circuit signature of M*. Accordingly, a *cocircuit signature of M* is a circuit signature of $M^\perp$, the *dual* (or *orthogonal*) *of M*.

THEOREM 2.2. *Let M be a matroid on a finite set E, let $\mathcal{O}$ be a circuit signature of M and let $\mathcal{O}'$ be a cocircuit signature of M.*

(a) *Then the following three properties are equivalent*:

(III) *for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$ such that $|\underline{X} \cap \underline{Y}| = 2$ or 3,*

$$(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing \quad \textit{and} \quad (X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing;$$

(IV) *for all $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$ such that $\underline{X} \cap \underline{Y} \neq \varnothing$,*

$$(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing \quad \textit{and} \quad (X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing;$$

(V) *for all $e \in E$ and all partitions of E into subsets R, G, B, W with $e \in R \cup G$, exactly one of the following holds*:

(i) *there exists $X \in \mathcal{O}$ such that*

$$e \in \underline{X} \subseteq R \cup G \cup B \quad \textit{and} \quad X^- \cap R = X^+ \cap G = \varnothing$$

or

(ii) *there exists $Y \in \mathcal{O}'$ such that*

$$e \in \underline{Y} \subseteq R \cup G \cup W \quad \textit{and} \quad Y^- \cup R = Y^+ \cap G = \varnothing.$$

(b) *Furthermore, $\mathcal{O}$ is an orientation of M if and only if there exists a cocircuit signature $\mathcal{O}^\perp$ of M such that for $\mathcal{O}' = \mathcal{O}^\perp$ the properties* (III), (IV), *and* (V) *are satisfied. In fact, if $\mathcal{O}$ is an orientation of M, then $\mathcal{O}^\perp$ is unique, and by symmetry $\mathcal{O}^\perp$ is an orientation of $M^\perp$.*

It is evident from Theorem 2.2 that a matroid is orientable if and only if its dual is orientable. Given an orientation $\mathcal{O}$ of M, the orientation $\mathcal{O}^\perp$ of $M^\perp$ described in part (b) of the theorem will be called the *dual* (*orthogonal*) of $\mathcal{O}$. Similarly $(E, \mathcal{O}^\perp)$ is called the *dual* (*orthogonal*) of $(E, \mathcal{O})$. Note that the uniqueness result in Theorem 2.2b implies that $(\mathcal{O}^\perp)^\perp = \mathcal{O}$, thus we speak of dual pairs of orientations and dual pairs of oriented matroids.

The properties (III), (IV), and (V) of Theorem 2.2 are related to conditions that Minty gave for digraphoids [12]. (That relationship is discussed in Section 6.) We will see in the next section that (III) and (IV) abstract the notion of orthogonality. Accordingly, signed sets X and Y having either $\underline{X} \cap \underline{Y} = \varnothing$, or $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing$ will be called *orthogonal*, and (IV) will be called the *orthogonality property of dual pairs of oriented matroids.*

In the remainder of this section we will, after briefly introducing some useful operations on matroid signatures, prove Theorems 2.1 and 2.2. The reader may wish to read Section 3, which provides examples and interpretations of oriented matroids, before reading these proofs.

Given X, a signed subset of E, and $A \subseteq E$, the signed set Z having $Z^+ = (X^+ \backslash A) \cup (X^- \cap A)$ and $Z^- = (X^- \backslash A) \cup (X^+ \cap A)$ is said to be obtained from X by *reversing signs* on A and is denoted by $Z = {}_{\bar{A}}X$. Thus $-X = {}_{\bar{E}}X$. For $\mathcal{O}$ a circuit signature of a matroid M on E and $A \subseteq E$, the circuit signature ${}_{\bar{A}}\mathcal{O}$ of M obtained from $\mathcal{O}$ by reversing signs on A is defined by ${}_{\bar{A}}\mathcal{O} = \{{}_{\bar{A}}X : X \in \mathcal{O}\}$.

Note that properties (I) and (II) of Theorem 2.1 are invariant under this operation. Similarly, properties (III), (IV), and (V) of Theorem 2.2 obviously hold for $\mathcal{O}$, $\mathcal{O}'$ if and only if they hold for ${}_{\bar{A}}\mathcal{O}$, ${}_{\bar{A}}\mathcal{O}'$ for all $A \subseteq E$.

Let $\mathcal{O}$ be a circuit signature of a matroid M on E and let $e \in E$. The set $\mathcal{O} \backslash e$ obtained by *deleting* e in $\mathcal{O}$ is defined by $\mathcal{O} \backslash e = \{X \in \mathcal{O} : e \notin \underline{X}\}$. Note that $\mathcal{O} \backslash e$ is a circuit signature of the matroid minor of M obtained by deleting e. In order to define the corresponding *contraction* operation we adopt the following notation. If X is a signed set, then $X \backslash e$ denotes the signed set Z having $Z^+ = X^+ \backslash e$ and $Z^- = X^- \backslash e$. For $\mathcal{O}$ a set of signed sets, define $\mathrm{Min}(\mathcal{O})$ to be the set of *minimal* members of $\mathcal{O}$, i.e., $\mathrm{Min}(\mathcal{O}) = \{X \in \mathcal{O} : X' \in \mathcal{O}$ and $\underline{X}' \subseteq \underline{X}$ imply $\underline{X}' = \underline{X}\}$. The set $\mathcal{O}/e$ obtained by *contracting* e *in* $\mathcal{O}$ is defined to be $\mathrm{Min}\{X \backslash e : X \in \mathcal{O}, X \backslash e \neq \varnothing\}$. Of course, $\mathcal{O}/e$ is a circuit signature of M/e, the matroid minor of M obtained by contracting e. The single element deletion and contraction operations described above will be very useful in the following proofs. The general subject of oriented matroid minors will be addressed directly in Section 4.

Proof of Theorem 2.1

It is clear that (II) implies (I). We will use the contraction and deletion operations to inductively prove that (I) implies (II).

Let $\mathcal{O}$ be a circuit signature of a matroid M on E and suppose that $\mathcal{O}$ satisfies (I). It is obvious that $\mathcal{O}|e$ also satisfies (I). In order to prove that $\mathcal{O}/e$ satisfies (I) we give two preliminary results.

LEMMA 2.1.1. *Let $X_1 \in \mathcal{O}$ with $X_1^- = \varnothing$ and let $X_2 \in \mathcal{O}$ have $\underline{X}_2 \backslash \underline{X}_1 = \{e\}$ with $e \in X_2^+$ and $X_2^- \neq \varnothing$. Then there is a signed circuit $X \in \mathcal{O}$ having $X^- = \varnothing$ and $(\underline{X}_1 \backslash X_2^-) + e \subseteq \underline{X}$.*

Proof. Let $x \in X_2^-$. By (I) there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+) \backslash x$ and $X_3^- \subseteq (X_1^- \cup X_2^-) \backslash x = X_2^- \backslash x$. It follows that $e \in \underline{X}_3$, otherwise $\underline{X}_3 \subsetneqq \underline{X}_1$, and since $e \in X_2^+ \backslash \underline{X}_1$, we have $e \in X_3^+$. Also

$$\underline{X}_1 \backslash \underline{X}_2 \subseteq \underline{X}_3, \tag{2.2}$$

otherwise by eliminating e from $\underline{X}_2$ and $\underline{X}_3$ we get a circuit of $\mathcal{O}$ properly contained in $\underline{X}_1$.

Now suppose that $y \in X_2^+ \backslash \underline{X}_3$. If we use (I) to eliminate e from X_2 and $-X_3$, then we get $X_1' \in \mathcal{O}$ having $\underline{X}_1' \subseteq \underline{X}_2 \cup \underline{X}_3 \backslash e \subseteq \underline{X}_1$, thus by (0) it must be that $X_1' = \pm X_1$. Note that $x, y \in X_1^+$, since $x, y \in \underline{X}_2$ and $X_1^- = \varnothing$. But $x \in X_2^- \backslash \underline{X}_3$, and $y \in X_2^+ \backslash \underline{X}_3$, so by (I) x and y do not agree in sign in X_1', a contradiction. Therefore $X_2^+ \subseteq \underline{X}_3$, so by (2.2) we have $(\underline{X}_1 \backslash X_2^-) + e \subseteq X_3^+$. If $X_3^- = \varnothing$, then the conclusion of the lemma is satisfied by $X = X_3$. Otherwise, we can repeat the argument above with X_3 in place of X_2. Thus we obtain $X_4 \in \mathcal{O}$ having $X_4^+ \supseteq (\underline{X}_1 \backslash X_3^-) + e \supseteq (\underline{X}_1 \backslash X_2^-) + e$ and $X_4^- \subsetneqq X_3^- \subsetneqq X_2^-$. The procedure can be repeated at most $|X_2^-|$ times until it terminates with a circuit $X_k \in \mathcal{O}$ satisfying $(\underline{X}_1 \backslash X_2^-) + e \subseteq \underline{X}_k$ and $X_k^- = \varnothing$.

LEMMA 2.1.2. *Let $X \in \mathcal{O}$ and $e \in E$. For all $x \in \underline{X} \backslash e$ there is a circuit $\hat{X} \in \mathcal{O}/e$ such that $x \in \underline{\hat{X}} \subseteq \underline{X} \backslash e$ and $\hat{X}^+ \subseteq X^+$, $\hat{X}^- \subseteq X^-$.*

Proof. By reversing signs on X^- we see that it suffices to establish the lemma in the case $X^- = \varnothing$. If $X \backslash e \in \mathcal{O}/e$, then obviously $\hat{X} = X \backslash e$ satisfies the conditions of the lemma. Suppose that $X \backslash e \notin \mathcal{O}/e$, so there exists $Z \in \mathcal{O}$ having $\varnothing \neq \underline{Z} \backslash e \subsetneqq \underline{X}$ and $e \in Z^+$. If $Z^- = \varnothing$, then set $Z_1 = Z$. If $Z^- \neq \varnothing$, then by Lemma 2.1.1 with $X_1 = X$ and $X_2 = Z$ there exists $Z_1 \in \mathcal{O}$ such that $Z_1^- = \varnothing$ and $e \in \underline{Z}_1 \subseteq \underline{X} + e$. By reversing e in $\mathcal{O}$, applying Lemma 2.1.1 with $X_1 = X$ and $X_2 = -Z_1$, then reversing e back again, we see that there is a $Z_2 \in \mathcal{O}$ such that $e \in \underline{Z}_2 \subseteq \underline{X} + e$, $Z_2^- \backslash e = \varnothing$ and $\underline{X} \backslash \underline{Z}_1 \subseteq \underline{Z}_2$. Now $Z_i \backslash e \in \mathcal{O}/e$ and $Z_i^- \backslash e = \varnothing$ for $i = 1, 2$, and $\underline{Z}_1 \cup \underline{Z}_2 = \underline{X}$, so $x \in \underline{Z}_1 \backslash e$ or $x \in \underline{Z}_2 \backslash e$.

LEMMA 2.1.3. *For any $e \in E$ both $\mathcal{O}/e$ and $\mathcal{O} \backslash e$ satisfy* (I).

Proof. It is clear that $\mathcal{O} \backslash e$ satisfies (I), since $\mathcal{O}$ satisfies (I). Let $\hat{X}_1$, $\hat{X}_2 \in \mathcal{O}/e$

with $\hat{X}_1 \neq -\hat{X}_2$ and $x \in \hat{X}_1^+ \cap \hat{X}_2^-$. There must exist $X_1, X_2 \in \mathcal{O}$ such that $\hat{X}_1 = X_1 \backslash e$, $\hat{X}_2 = X_2 \backslash e$, hence $X_1 \neq -X_2$ and $x \in X_1^+ \cap X_2^-$. By (I) we get $X_3 \in \mathcal{O}$ having $X_3^+ \subseteq (X_1^+ \cup X_2^+) \backslash x$ and $X_3^- \subseteq (X_1^- \cup X_2^-) \backslash x$. Lemma 2.1.2 implies that there is an $\hat{X}_3 \in \mathcal{O}/e$ satisfying $\hat{X}_3^+ \subseteq (\hat{X}_1^+ \cup \hat{X}_2^+) \backslash x$, $\hat{X}_3^- \subseteq (\hat{X}_1^- \cup \hat{X}_2^-) \backslash x$.

Now we can establish that $\mathcal{O}$ satisfies (II). This is trivial when $|E| = 1$; suppose that it holds whenever $|E| \leqslant p$. Let $|E| = p + 1 \geqslant 2$. Note that the inductive hypothesis and Lemma 2.1.3 imply that $\mathcal{O}/e$ and $\mathcal{O}\backslash e$ satisfy (II) for all $e \in E$. Let $X_1, X_2 \in \mathcal{O}$ have the smallest possible value of $|\underline{X}_2 \backslash \underline{X}_1|$ subject to the existence of elements $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ and $y \in (X_1^+ \backslash X_2^-) \cup (X_1^- \backslash X_2^+)$ such that there is no $X_3 \in \mathcal{O}$ satisfying

$$X_3^+ \subseteq (X_1^+ \cup X_2^+) \backslash x, \qquad X_3^- \subseteq (X_1^- \cup X_2^-) \backslash x, \qquad \text{and} \qquad y \in \underline{X}_3. \tag{2.3}$$

Since properties (I) and (II) are invariant under reversal of signs on a subset of E, there is no loss of generality in assuming that $X_1^- = \varnothing$, $X_2^- \backslash \underline{X}_1 = \varnothing$. Note that if $|\underline{X}_2 \backslash \underline{X}_1| = 1$, then Lemma 2.1.1 implies that (2.3) can be satisfied by some $X_3 \in \mathcal{O}$, hence $|\underline{X}_2 \backslash \underline{X}_1| \geqslant 2$.

Suppose that $e \in X_2^- \backslash x$, implying that $e \in X_1^+$. Then $\hat{X}_1 = X_1 \backslash e$ and $\hat{X}_2 = X_2 \backslash e$ are in $\mathcal{O}/e$, which satisfies (II). Thus there is some $\hat{X}_3 \in \mathcal{O}/e$ such that $\hat{X}_3^+ \subseteq (\hat{X}_1^+ \cup \hat{X}_2^+) \backslash x$, $\hat{X}_3^- \subseteq (\hat{X}_1^- \cup \hat{X}_2^-) \backslash x$, and $y \in \underline{\hat{X}}_3$. Now the circuit $X_3 \in \mathcal{O}$ having $\hat{X}_3 = X_3 \backslash e$ satisfies (2.3), since the sign of e is not constrained by (2.3). So we may assume that $X_2^- = \{x\}$, and (2.3) reduces to

$$X_3^- = \varnothing, \qquad \underline{X}_3 \subseteq (\underline{X}_1 \cup \underline{X}_2) \backslash x, \qquad \text{and} \qquad y \in \underline{X}_3. \tag{2.4}$$

Let $e \in \underline{X}_2 \backslash \underline{X}_1$. By Lemma 2.1.2 there is an $\hat{X}_1 \in \mathcal{O}/e$ such that $\hat{X}_1^- = \varnothing$ and $y \in \underline{\hat{X}}_1 \subseteq \underline{X}_1$. If $x \notin \underline{X}_1$, set $\hat{Z} = \hat{X}_1$. If $x \in \underline{\hat{X}}_1$, since $\hat{X}_2 = X_2 \backslash e \in \mathcal{O}/e$, which satisfies (II), there is a nonnegative $\hat{Z} \in \mathcal{O}/e$ such that $y \in \underline{\hat{Z}} \subseteq (\underline{\hat{X}}_1 \cup \hat{X}_2) \backslash x$. Let $Z \in \mathcal{O}$ have $\hat{Z} = Z \backslash e$. If $Z^- = \varnothing$, then (2.4) is satisfied by $X_3 = Z$, so $Z^- = \{e\}$. Since $e \in X_2^+$, we can apply (I) to establish the existence of $X_2' \in \mathcal{O}$ such that $X_2'^+ \subseteq (X_2^+ \cup Z^+) \backslash e$ and $X_2'^- \subseteq (X_2^- \cup Z^-) \backslash e = \{x\}$.

Now there are two cases to consider:

(i) Suppose that $x \notin X_2'^-$, which implies $X_2'^- = \varnothing$. If $y \in \underline{X}_2'$, then $X_3 = X_2'$ satisfies (2.4), so assume that $y \notin \underline{X}_2'$. Now $x \in \underline{X}_1 \backslash \underline{X}_2'$, so there is an element $e' \in \underline{X}_2' \backslash \underline{X}_1$, and $e' \neq e$ since $e \notin \underline{X}_2'$. By repeating the arguments above for $\mathcal{O}/e'$, rather than $\mathcal{O}/e$, we either construct an $X_3 \in \mathcal{O}$ satisfying (2.4), or a $Z' \in \mathcal{O}$ such that $y \in \underline{Z}' \subseteq (\underline{X}_1 \cup \underline{X}_2) \backslash x$ and $Z'^- = \{e'\}$. However, Z', $X_2' \in \mathcal{O}\backslash x$, which satisfies (II) by the inductive hypothesis. Note that $\{e'\} = (Z'^- \cap X_2'^+) \cup (Z'^+ \cap X_2'^-)$, $\underline{Z}' \cup \underline{X}_2' \subseteq \underline{X}_1 \cup \underline{X}_2$, and $y \in Z'^+ \backslash \underline{X}_2'$. So applying (II) with $X_1 = Z'$ and $X_2 = X_2'$ gives an X_3 that satisfies (2.4).

(ii) If $x \in X_2'^-$, then $X_2'^- = \{x\}$, so certainly $X_2' \neq \pm X_1$. But since

$\underline{X}_2' \subseteq (\underline{X}_2 \cup \underline{Z})\backslash e \subseteq (\underline{X}_1 \cup \underline{X}_2)\backslash e$, we have $\underline{X}_2'\backslash \underline{X}_1 \subseteq (\underline{X}_2\backslash \underline{X}_1)\backslash e$. Therefore, by the choice of X_1 and X_2, the elimination property (II) must hold for X_1, X_2'. In particular since $x \in X_1^+ \cap X_2'^-$ and $y \in X_1^+\backslash X_2'^-$, there exists X_3 satisfying $X_3^+ \subseteq (X_1^+ \cup X_2'^+)\backslash x$, $X_3^- \subseteq (X_1^- \cup X)_2'^-\backslash x$, and $y \in \underline{X}_3$. Since $X_2'^- = \{x\}$, $X_1^- = \varnothing$, we see that $X_3^- = \varnothing$. Moreover, $\underline{X}_2' \subseteq \underline{X}_2 \cup \underline{Z} \subseteq \underline{X}_2 \cup \underline{X}_1$ so $\underline{X}_3 \subseteq (\underline{X}_1 \cup \underline{X}_2)\backslash x$, and (2.4) is satisfied by X_3. Thus Theorem 2.1 is established.

Proof of Theorem 2.2

This proof is broken into several parts. First we establish the equivalence of (IV) and (V) in part (a) of the theorem. We refer to a partition R, G, B, W of E with $R \cup G \neq \varnothing$, as in (V), as a *4-painting* of E into red, green, blue, and white elements.

Proof of (IV) $\Rightarrow$ (V). Assume given a 4-painting R, G, B, W of E and a distinguished element $e \in R \cup G$.

Suppose that $X \in \mathcal{O}$ satisfies alternative (i) of (V) and $Y \in \mathcal{O}'$ satisfies alternative (ii) of (V). Then $e \in (X^+ \cap Y^+) \cup (X^- \cap Y^-)$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$, a contradiction. Hence alternatives (V.i) and (V.ii) cannot both hold. We will now show by induction on the cardinality of $R \cup G$ that at least one of (V.i) and (V.ii) must hold.

Suppose that $|R \cup G| = 1$, i.e., $R \cup G = \{e\}$, and (V.i) fails. Then e is not in the closure of B, so there is a hyperplane H of M such that $B \subseteq H$ and $e \notin H$. Thus for some $Y \in \mathcal{O}'$ we have $e \in \underline{Y} = E\backslash H \subseteq W + e$ and either Y or $-Y$ satisfies (V.ii).

Now assume that the result holds for all 4-paintings having no more than p red and green elements, where $p \geqslant 1$, and that it fails for the 4-painting R, G, B, W with $e \in R \cup G$ the distinguished element and $|R \cup G| = p + 1 \geqslant 2$.

Select $e' \in R \cup G$, $e' \neq e$, and let R', G', B', W' and R'', G'', B'', W'', respectively, be the 4-paintings obtained from R, G, B, W by repainting e' first blue and then white. Since $|R' \cup G'| = p$, either (V.i) or (V.ii) is satisfied with respect to R', G', B', W' and $e \in R' \cup G'$. But a $Y \in \mathcal{O}'$ satisfying (V.ii) for this painting would also satisfy (V.ii) for the original painting, a contradiction. Hence (V.i) is satisfied by some $X \in \mathcal{O}$ having $e \in \underline{X} \subseteq R' \cup G' \cup B'$, $X^- \cap R' = X^+ \cap G' = \varnothing$. Furthermore, since (V.i) fails for the original painting, $e' \in (X^- \cap R) \cup (X^+ \cap G)$. Similarly, since $|R'' \cup G''| = p$, we know that there is a $Y \in \mathcal{O}'$ such that $e \in \underline{Y} \subseteq R'' \cup G'' \cup W''$, $Y^- \cap R'' = Y^+ \cap G'' = \varnothing$, and $e' \in (Y^- \cap R) \cup (Y^+ \cap G)$. But then $\{e, e'\} \subseteq (X^+ \cap Y^+) \cup (X^- \cap Y^-)$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$, a contradiction of the orthogonality condition (IV).

Proof of (V) $\Rightarrow$ (IV). Suppose that $\mathcal{O}, \mathcal{O}'$ satisfies (V) and that $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ with X and Y not orthogonal. Replacing Y by $-Y$, if necessary, we can assume that $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$. Let $R = X^+ \cup Y^+$, $G = E \backslash R \supseteq X^- \cup Y^-$, $B = W = \varnothing$, and distinguish any $e \in (X^+ \cap Y^+) \cup (X^- \cap Y^-)$. Then X satisfies alternative (V.i) and Y satisfies (V.ii), a contradiction.

It is clear that property (IV) implies (III). Before completing the proof of part (a) of Theorem 2.2 by showing that (III) implies (IV), it will be useful to prove the following lemma, which establishes one of the implications in part (b) of the theorem.

LEMMA 2.2.1. *If $\mathcal{O}$, $\mathcal{O}'$ is a pair of circuit and cocircuit signatures of a matroid and $\mathcal{O}$, $\mathcal{O}'$ satisfies* (V), *then each of $\mathcal{O}$ and $\mathcal{O}'$ satisfies* (I) *and is, therefore, a matroid orientation.*

Proof. By symmetry, it is enough to show that $\mathcal{O}$ satisfies (I). Let $X_1, X_2 \in \mathcal{O}$, with $X_1 \neq -X_2$ and $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$. Consider the following 4-painting of E:

$$R = (X_1^+ \backslash X_2^-) \cup (X_2^+ \backslash X_1^-), \qquad G = (X_1^- \backslash X_2^+) \cup (X_2^- \backslash X_1^+),$$
$$B = [(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)] \backslash x, \qquad W = [E \backslash (\underline{X}_1 \cup \underline{X}_2)] + x,$$

and distinguish any $e \in \underline{X}_1 \backslash \underline{X}_2 \subseteq R \cup G$. Suppose that $Y \in \mathcal{O}'$ satisfies alternative (V.ii) with respect to this 4-painting. Then $e \in (X_1^+ \cap Y^+) \cup (X_1^- \cap Y^-)$ and $(X_1^+ \cap Y^-) \cup (X_1^- \cap Y^+) \subseteq \{x\}$. Since the equivalence of (IV) and (V) has been established and the pair $\mathcal{O}$, $\mathcal{O}'$ satisfies (V), it must also satisfy (IV), implying that

$$x \in (X_1^+ \cap Y^-) \cup (X_1^- \cap Y^+). \tag{2.5}$$

But $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ so by (2.5) $x \in (X_2^- \cap Y^-) \cup (X_2^+ \cap Y^+)$, yet $(X_2^+ \cap Y^-) \cup (X_2^- \cap Y^+) = \varnothing$, a contradiction. So alternative (V.ii) fails and (V.i) must hold. This implies the existence of some $X \in \mathcal{O}$ having $\underline{X} \subseteq R \cup G \cup B = (\underline{X}_1 \cup \underline{X}_2) \backslash x$ and $X^+ \subseteq R \cup B \subseteq X_1^+ \cup X_2^+$, $X^- \subseteq G \cup B \subseteq X_1^- \cup X_2^-$. Therefore the signed elimination property (I) is satisfied by $\mathcal{O}$. Since $\mathcal{O}$ is a matroid signature, it also satisfies (0) of Theorem 2.1, hence $\mathcal{O}$ is a matroid orientation.

Note that this proof, with no further work, indicates directly that $\mathcal{O}$ and $\mathcal{O}'$ satisfy the stronger elimination property (II).

We will now use Lemma 2.2.1 to complete the proof of part (a) of Theorem 2.2 by showing that (III) implies (IV).

Proof of (III) $\Rightarrow$ (IV). Let $\mathcal{O}$, $\mathcal{O}'$ satisfy (III). Note that for any choice of $e \in E$ it follows that $\mathcal{O}/e$, $\mathcal{O}'\backslash e$ and $\mathcal{O}\backslash e$, $\mathcal{O}'/e$ also satisfy (III). For $|E|$ sufficiently small the result must hold. Suppose that it fails for the pair $\mathcal{O}$, $\mathcal{O}'$, but holds for all pairs of matroid signatures on fewer than $|E|$ elements. Let $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$ with X and Y not orthogonal. We assume without loss of generality that $(X^+ \cap Y^-) \cup (X^- \cap Y^+) = \varnothing$, since if that is not the case for X and Y, it is the case for X and $-Y$. Furthermore, we can reverse signs in $\mathcal{O}$ and $\mathcal{O}'$ on $X^- \cup Y^-$, since (III) and (IV) are invariant under such reversals, so we may assume that $X^- = Y^- = \varnothing$.

Suppose that $e \in \underline{X} \backslash \underline{Y}$. Then $X \backslash e \in \mathcal{O}/e$ and $Y \in \mathcal{O}' \backslash e$. But $X \backslash e$ and Y are not orthogonal, yet the pair $\mathcal{O}/e$, $\mathcal{O}' \backslash e$ satisfies (III) and, by the inductive hypothesis, (IV). Thus $\underline{X} \subseteq \underline{Y}$. Similarly $\underline{Y} \subseteq \underline{X}$, so $\underline{X} = \underline{Y}$.

Suppose $u, v \in E \backslash \underline{X}$, $u \neq v$. Since $Y \in \mathcal{O}' \backslash u$ and $\mathcal{O}/u$, $\mathcal{O}' \backslash u$ satisfies (III), and hence (IV), it follows that $X \backslash u \notin \mathcal{O}/u$. Thus there is some $U \in \mathcal{O}$ such that

$$\{u\} \subsetneqq \underline{U} \subsetneqq \underline{X} + u. \tag{2.6}$$

Select U so that $|U^- \cap \underline{Y}|$ is minimized subject to (2.6). Note that $U^- \cap \underline{Y} \neq \varnothing$ since $U \backslash u \in \mathcal{O}/u$ must be orthogonal to $Y \in \mathcal{O}' \backslash u$. Let $w \in U^- \cap \underline{Y}$ and observe that $U, X \in \mathcal{O} \backslash v$. Now $\mathcal{O} \backslash v$, $\mathcal{O}'/v$ satisfies (III) and, by the inductive hypothesis, it satisfies (IV). By Lemma 2.2.1 $\mathcal{O} \backslash v$ satisfies (I) and, therefore, (II). Hence there exists $\hat{V} \in \mathcal{O} \backslash v$ such that $\hat{V}^+ \subseteq (X^+ \cup U^+) \backslash w$ and $\hat{V}^- \subseteq (X^- \cup U^-) \backslash w$. Let $V \in \mathcal{O}$ such that $\hat{V} = V \backslash v$ and observe that $\underline{V} \subseteq (\underline{X} \cup \underline{U}) \backslash w \subseteq \underline{X} + u \backslash w$ so $u \in \underline{V}$. But then $\{u\} \subsetneqq \underline{V} \subsetneqq \underline{X} + u$ and $V^- \cap \underline{Y} = V^- \cap \underline{X} \subsetneqq U^- \cap \underline{X}$, contradicting the choice of U. Hence there exist no distinct u and v in $E \backslash \underline{X}$, implying that $|E| \leqslant |\underline{X}| + 1 = |\underline{Y}| + 1$. If r is the Whitney rank of the matroid $(E, \underline{\mathcal{O}})$ then $|\underline{X}| \leqslant r + 1$ and $|\underline{Y}| \leqslant |E| - r + 1$ since $\underline{X} \in \underline{\mathcal{O}}$ and $\underline{Y} \in \underline{\mathcal{O}}^\perp$. Therefore $|E| \leqslant |\underline{Y}| + 1 \leqslant |E| - r + 2$ so $r \leqslant 2$ and $|E| \leqslant |\underline{X}| + 1 \leqslant r + 2 \leqslant 4$. So, $|\underline{X} \cap \underline{Y}| \leqslant 3$ and orthogonality of X and Y follows from (III).

The following example indicates that if (III) is relaxed to require orthogonality only for those $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ having $|\underline{X} \cap \underline{Y}| = 2$, then (IV) is no longer implied. Let M be the four-point line, the self-dual matroid on a four-element set E having as its circuits the four triples in E. Let $\mathcal{O}$ and $\mathcal{O}'$ both be given by the rows and the opposites of the rows in the following 4×4 array.

$$\begin{array}{cccc}
e_1 & e_2 & e_3 & e_4 \\
+ & + & + & 0 \\
+ & - & 0 & + \\
+ & 0 & - & - \\
0 & + & - & +
\end{array}$$

Note that $\mathcal{O}$, $\mathcal{O}'$ does not satisfy (IV) (no oriented matroid can be self-dual), but orthogonality is satisfied for all $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ having $|\underline{X} \cap \underline{Y}| = 2$.

To complete the proof of Theorem 2.2 it suffices to establish

LEMMA 2.2.2. *If $M = (E, \mathcal{O})$ is an oriented matroid, then there is a unique cocircuit signature $\mathcal{O}^\perp$ of $\underline{M}$ such that $\mathcal{O}$, $\mathcal{O}^\perp$ satisfies the orthogonality condition* (IV).

To prove Lemma 2.2.2. we first recall a familiar property of matroids.

LEMMA 2.2.3. *Let $M = (E, \mathscr{C})$ be a matroid. For any $C \in \mathscr{C}$ and $e, e' \in C$, $e \neq e'$, there exists $D \in \mathscr{C}^\perp$ such that $C \cap D = \{e, e'\}$.*

Proof. The set $C \backslash e$ is independent in M, so $(E \backslash C) + e$ contains a dual base $B^\perp$. Therefore there is a cocircuit $D \subseteq B^\perp + e' \subseteq (E \backslash C) \cup \{e, e'\}$ and $e' \in D$. Now $e' \in C \cap D \subseteq \{e, e'\}$, so $e \in C \cap D$, otherwise $| C \cap D | = 1$.

From Lemma 2.2.3 we see that for any circuit signature $\mathcal{O}$ of a matroid M there exists a cocircuit signature $\mathcal{O}'$ of M satisfying the condition

> for every $Y \in \mathcal{O}'$ there exists an element $e \in \underline{Y}$ such that for all $y \in \underline{Y}$, $y \neq e$, there is some $X \in \mathcal{O}$ having $\underline{X} \cap \underline{Y} = \{e, y\}$ and X orthogonal to Y. (2.7)

Proof of Lemma 2.2.2. Let $\mathcal{O}'$ be any cocircuit signature of $\underline{M}$ satisfying (2.7). Suppose that $X \in \mathcal{O}$, $Y \in \mathcal{O}'$, X and Y are not orthogonal, and $| \underline{X} \cap \underline{Y} |$ is as small as possible, subject to the conditions above. Since X and Y are not orthogonal, $\underline{X} \cap \underline{Y} \neq \varnothing$ and thus $| \underline{X} \cap \underline{Y} | \geqslant 2$, because $\underline{X}$ is a circuit and $\underline{Y}$ is a cocircuit of the matroid $\underline{M}$. Let $x, y \in \underline{X} \cap \underline{Y}$, $x \neq y$. By reversing signs in $\mathcal{O}$ and $\mathcal{O}'$ and replacing X or Y by its opposite, if necessary, we can assume that $X^- = Y^- = \varnothing$. Thus, $x, y \in X^+ \cap Y^+$.

We will first show that there is a signed circuit $Z \in \mathcal{O}$ such that

$$x \in Z^+, \qquad y \in Z^-, \qquad \text{and } \underline{Z} \cap \underline{Y} = \{x, y\}. \tag{2.8}$$

Now $Y \in \mathcal{O}'$ and the pair $\mathcal{O}, \mathcal{O}'$ satisfies (2.7). Hence there is an $e \in \underline{Y}$ such that for each $z \in \underline{Y}$, $z \neq e$, there exists $X_z \in \mathcal{O}$ having $\underline{X}_z \cap \underline{Y} = \{e, z\}$ and X_z orthogonal to Y. If $e = x$, then either $Z = X_y$ or $Z = -X_y$ satisfies (2.8), and if $e = y$ then $Z = X_x$ or $Z = -X_x$ satisfies (2.8). Suppose that $e \neq x, y$. Then, replacing X_x or X_y by its opposite, if necessary, we have

$$e \in X_x^- \cap X_y^+, \qquad x \in X_x^+ \backslash \underline{X}_y, \qquad y \in X_y^- \backslash \underline{X}_x,$$

and

$$(\underline{X}_x \cup \underline{X}_y) \cap \underline{Y} \subseteq \{x, y, e\}.$$

By (II) there is a $Z \in \mathcal{O}$ with $Z^+ \subseteq (X_x^+ \cup X_y^+) \backslash e$, $Z^- \subseteq (X_x^- \cup X_y^-) \backslash e$, and $x \in Z^+$. So $x \in \underline{Z} \cap \underline{Y} \subseteq \{x, y\}$, thus $y \in Z^-$ and (2.8) is satisfied.

Now we have $x \in X^+ \cap Z^+$ and $y \in X^+ \cap Z^-$. By (II) there exists a signed circuit $X' \in \mathcal{O}$ with $x \in X'^{+} \subseteq (X^{+} \cup Z^{+}) \backslash y$ and $X'^{-} \subseteq (X^{-} \cup Z^{-}) \backslash y$. Now $X^{-} \cap \underline{Y} = \varnothing$ and $Z^{-} \cap \underline{Y} = \{y\}$ so $X'^{-} \cap \underline{Y} = \varnothing$, implying that $X' \in \mathcal{O}$ is not orthogonal to Y, since $x \in X'^{+} \cap Y^{+}$ and $Y^{-} = \varnothing$. Moreover, $\underline{X}' \cap \underline{Y} \subsetneq \underline{X} \cap \underline{Y}$, contradicting the choice of X. Therefore, all $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ are orthogonal, so (IV) is satisfied by $\mathcal{O}$, $\mathcal{O}'$, i.e., $\mathcal{O}'$ is *dual to* $\mathcal{O}$. Moreover, by Lemma 2.2.3 there can be at most one cocircuit signature $\mathcal{O}'$ of $\underline{M}$ that has X and Y orthogonal for all $X \in \mathcal{O}$, $Y \in \mathcal{O}'$ such that $|\underline{X} \cap \underline{Y}| = 2$. Hence $\mathcal{O}'$ is the unique cocircuit signature of $\underline{M}$ that is dual to $\mathcal{O}$. This completes the proof of Theorem 2.2.

3. Examples

EXAMPLE 3.1. Oriented matroids coordinatizable over an ordered field. Let F be an ordered field let E be a finite set, and let $\mathscr{R}$ be a vector subspace of F^{E}. Consider the set $\mathcal{O}$ of signed supports of elementary vectors of $\mathscr{R}$ and the set $\mathcal{O}'$ of signed supports of elementary vectors of $\mathscr{R}^{\perp}$, the orthogonal complement of $\mathscr{R}$. Clearly $\mathcal{O}$ is a circuit signature and $\mathcal{O}'$ is a cocircuit signature of the matroid $(E, \underline{\mathcal{O}})$. If $X \in \mathcal{O}$ and $Y \in \mathcal{O}'$, then there are elementary vectors $\alpha \in \mathscr{R}$ and $\beta \in \mathscr{R}^{\perp}$ such that $X^{+} = S^{+}(\alpha)$, $X^{-} = S^{-}(\alpha)$ and $Y^{+} = S^{+}(\beta)$, $Y^{-} = S^{-}(\beta)$. It follows that X and Y are orthogonal as signed sets, since α and β are orthogonal vectors in F^{E}. Thus the orthogonality property of Theorem 2.2 is satisfied by $\mathcal{O}$, $\mathcal{O}'$, $(E, \mathcal{O})$ is an oriented matroid and $\mathcal{O}^{\perp} = \mathcal{O}'$; we denote by $S(\mathscr{R})$ the oriented matroid $(E, \mathcal{O})$.

An oriented matroid $M = (E, \mathcal{O})$ that arises in this way is said to be *coordinatizable* (or *representable*) over F. If, for a given ordering of E, A is an $m \times n$ matrix over F with $\mathscr{R}$ as its null space, then A is called a coordinatization of M. (More properly, we might call A a *Whitney coordinatization on of* M and a *Tutte coordinatization of* $M^{\perp}$.) In this case E can be considered to be the family $\{e_1 = a_1, \ldots, e_n = a_n\}$ of points in F^{m}, where $a_1, \ldots, a_n$ are the columns of A, and we say that M is *the oriented matroid on E determined by linear dependence in* F^{m}.

Example 3.1 yields

PROPOSITION 3.2. *All matroids coordinatizable over an ordered field are orientable.*

For additional generality in Example 3.1, and Proposition 3.2, we could let F be noncommutative, i.e., an ordered division ring, and let $\mathscr{R}$ be a left (right) vector subspace of the left (right) vector space F^{E}. In fact, the reader familiar with [4] will recognize that Example 3.1 generalizes when F is an ordered unitary ring and $\mathscr{R}$ is a *unimodular module* (see [4]). Thus, for example, any integral chain group $\mathscr{R}$ describes an oriented matroid.

EXAMPLE 3.3. Graphic oriented matroids.

Let A be the $(0, \pm 1)$-vertex-edge incidence matrix of a directed graph $\Gamma = (V, E)$ and let $M = (E, \mathcal{O})$ be the oriented matroid coordinatized by A. Then $\underline{\mathcal{O}}(\underline{\mathcal{O}}^{\perp})$ is the collection of edge sets of elementary circuits (cocircuits) in Γ. If $X \in \mathcal{O}$ and the corresponding circuit is traversed so that some $e \in X^+$ is encountered as a forward edge or some $e \in X^-$ is encountered as a reverse edge, then the set of all forward (reverse) edges so encountered will be $X^+(X^-)$. For $Y \in \mathcal{O}^{\perp}$, removal of the edges of $\underline{Y}$ *cuts* a previously connected component of Γ into two connected components, with every edge of $\underline{Y}$ having one vertex in each. Then Y^+ is the subset of $\underline{Y}$ crossing the cut in one direction and Y^- consists of those edges of $\underline{Y}$ crossing the cut in the opposite direction.

Hence the following proposition, which follows immediately from Theorem 2.2 with $G = \varnothing$ in (V), is a generalization of Minty's painting lemma for directed graphs (see [12]).

PROPOSITION 3.4. *Let $M = (E, \mathcal{O})$ be an oriented matroid. Distinguish an element $e \in E$ and partition E into subsets $e \in R$, B, W. Then exactly one of the following alternatives holds*:

(i) *there is a signed circuit $X \in \mathcal{O}$ having $e \in \underline{X} \subseteq R \cup B$ and $X^- \cap R = \varnothing$; or*

(ii) *there is a signed cocircuit $Y \in \mathcal{O}$ having*

$$e \in \underline{Y} \subseteq R \cup W \qquad \textit{and} \qquad Y^- \cap R = \varnothing.$$

Minty's extension of his painting lemma from directed graphs to digraphoids [12] is the special case of Proposition 3.4 for binary oriented matroids (as we shall see in Section 6). Camion [4], Fulkerson [8], and Rockafellar [13] extended the result further, to the case of oriented matroids coordinatizable over an ordered field.

EXAMPLE 3.5. Affine coordinatizations of oriented matroids.

Let A be an $m \times n$ matrix over an ordered field F and let $M = (E, \mathcal{O})$ be the oriented matroid coordinatized by A. Let $\hat{A}$ be the $(m + 1) \times n$ matrix over F obtained from A by adding as a row the vector $(1,..., 1) \in F^n$ and let $\hat{M} = (E, \mathcal{O})$ be the oriented matroid coordinatized by $\hat{A}$. We say that A is an *affine* coordinatization of $\hat{M}$. Think of E as the family of points in F^m described by the columns of A. Then $\mathcal{O}$ is the set of signed supports of elementary vectors of the subspace $\{\alpha \in F^E : \sum_{e \in E} \alpha(e)\, e = 0 \text{ and } \sum_{e \in E} \alpha(e) = 0\}$ of F^E; we say that $\hat{M}$ is *the oriented matroid on E determined by affine dependence over F.*

We call a matroid orientation $\mathcal{O}$ that arises as in Example 3.1 (or 3.5) a canonical orientation of $(E, \mathcal{O})$. In Example 3.3 we saw that a canonical orientation that is induced by the $(0, \pm 1)$-vertex-edge incidence matrix of a directed graph has a simple graphical interpretation. We will now give a general geometric interpretation of canonical matroid orientations.

Let F be an ordered field, let m be a positive integer, let E be a finite family of points in F^m, and let $\mathcal{O}$ be the canonical matroid orientation determined by linear dependence over F in E. Recall that $\mathcal{O}$ is the set of signed supports of elementary vectors of the subspace $\mathcal{R} \subseteq F^E$ consisting of all $\alpha \in F^E$ having $\sum_{e \in E} \alpha(e)\, e$ equal to the zero vector in F^m. Let $X \in \mathcal{O}$. If $|\underline{X}| = 1$, then the subset $\{X, -X\} \subseteq \mathcal{O}$ can be trivially described. Suppose that $|\underline{X}| \geqslant 2$. Then for some elementary vector $\alpha \in \mathcal{R}$ we have $X = (S^+(\alpha), S^-(\alpha))$ and $\sum_{e \in \underline{X}} \alpha(e)\, e = 0$. Let $x, y \in \underline{X}$, $x \neq y$, so $\alpha(x) \neq 0$, $\alpha(y) \neq 0$. If $\alpha(x)$ and $\alpha(y)$ have the same sign, then $\alpha(x) + \alpha(y) \neq 0$ so we have

$$[\alpha(x) + \alpha(y)]^{-1}\, [\alpha(x)\, x + \alpha(y)\, y] \\ = -\, [\alpha(x) + \alpha(y)]^{-1} \Big[\sum_{e \in \underline{X} \setminus \{x, y\}} \alpha(e)\, e \Big]. \tag{3.1}$$

In other words, if $\alpha(x)$ and $\alpha(y)$ have the same sign, then the vector subspace of F^m generated by $\underline{X} \setminus \{x, y\}$ intersects the line segment between x and y. (We adopt the convention that the subspace generated by the empty set consists of the zero vector.) The converse can also be easily verified.

Having characterized $\mathcal{O}$ as above, we can give a geometric characterization of $\mathcal{O}^\perp$. Recall that the cocircuits of a matroid are the complements of hyperplanes. Assume that the rank of E in F^m, i.e., the rank of $\underline{M}$, is $r \leqslant m$. Then the hyperplanes of $\underline{M}$ correspond to the $(r - 1)$-dimensional subspaces of F^m generated by independent subsets of E; of course, if $r = m$ then these $(r - 1)$-dimensional subspaces are hyperplanes in F^m. It follows from Lemma 2.2.3, orthogonality, and the characterization of $\mathcal{O}$ above that if $Y \in \mathcal{O}^\perp$ and $u, v \in \underline{Y}$, $u \neq v$, then u and v have the same sign in Y if and only if they are on the same side of the vector subspace of F^m generated by $E \setminus \underline{Y}$.

These geometric interpretations remain interesting when the notion of linear dependence is replaced by affine dependence. In the case when M is determined by affine dependence, we have $\sum_{e \in \underline{X}} \alpha(e) = 0$ in (3.1). Thus (3.1) can be rewritten as

$$[\alpha(x) + \alpha(y)]^{-1}\, [\alpha(x)\, x + \alpha(y)\, y] \\ = \Big[\sum_{e \in \underline{X} \setminus \{x, y\}} \alpha(e) \Big]^{-1} \Big[\sum_{e \in \underline{X} \setminus \{x, y\}} \alpha(e)\, e \Big].$$

Therefore we have

PROPOSITION 3.6. *Let F be an ordered field and let E be a finite family of points in F^m. Suppose that $M = (E, \mathcal{O})$ is the oriented matroid on E determined by linear (affine) dependence over F. For $X \in \mathcal{O}$ and $x, y \in \underline{X}$, $x \neq y$, x and y*

agree in sign in X if and only if the linear (affine) subspace of F^m *generated by* $\underline{X}\backslash\{x, y\}$ *intersects the line segment between x and y. Furthermore, for* $Y \in \mathcal{O}^{\perp}$ *and* $u, v \in \underline{Y}$, $u \neq v$, *u and v have the same sign in Y if and only if u and v are on the same side of the linear (affine) subspace of* F^m *generated by* $E\backslash\underline{Y}$.

EXAMPLE 3.7. Minors of the Möbius geometry of Cheung and Crapo.

Let E be a finite subset of $\mathbb{R}^2$ and let $\mathscr{C}$ be the set of all subsets of E consisting of either four points on a common line, four points on a common circle, or five points with no four on a common line or circle. Then $M = (E, \mathscr{C})$ is a matroid minor of the Möbius geometry introduced by Cheung and Crapo [5].

M is coordinatizable over the reals. Let $f: E \to \mathbb{R}^4$ be defined by $f(a, b) = (a^2 + b^2, a, b, 1)$ for $(a, b) \in E$. Then a subset $T \subseteq E$ is dependent in M if and only if $f(T) = \{f(e) : e \in T\}$ is linearly dependent in $\mathbb{R}^4$. Hence by Proposition 3.2 M is orientable.

The canonical orientation $\mathcal{O}$ of M induced by f, which can be interpreted in $\mathbb{R}^4$ as in Proposition 3.6, has an interesting interpretation in $\mathbb{R}^2$. Hyperplanes of M are intersections of E with circles and lines. Each triple in E determines a hyperplane H that partitions $E\backslash H$ into two subsets (interior and exterior points of a circle H, or the sets of points on either side of a line H), which form the positive and negative elements of the cocircuit $E\backslash H$. It follows from orthogonality that if $X \in \mathcal{O}$ has $|\underline{X}| = 4$, then the signs in X of the elements of $\underline{X}$ alternate along the line or circle that they define. Suppose that $X \in \mathcal{O}$ has $|\underline{X}| = 5$ and $x, y \in \underline{X}$, $x \neq y$. Then $\underline{X}\backslash\{x, y\}$ defines a line or circle H, and x and y have the same sign in X if and only if the line segment joining them crosses H.

EXAMPLE 3.8. Alternating orientations.

A circuit signature $\mathcal{O}$ of a matroid $(E, \underline{\mathcal{O}})$ is said to be alternating with respect to an order $H: e_1 < e_2 < \cdots < e_n$ of the elements of E if for every $X \in \mathcal{O}$ with, say, $\underline{X} = \{e_{i_1}, \ldots, e_{i_s}\}$, $i_1 < i_2 < \cdots < i_s$, $sg_X(e_{i_{j+1}}) = -sg_X(e_{i_j})$, $j = 1, \ldots, s - 1$.

PROPOSITION 3.9. *Let the elements of E be denoted* $e_1, \ldots, e_n$ *and let H be the order* $e_1 < e_2 < \cdots < e_n$. *If* $M = (E, \mathscr{C})$ *is a matroid with the property*

$$\text{for all } C \in \mathscr{C},\ D \in \mathscr{C}^{\perp} \text{ with } |C \cap D| = 2 \text{ or } 3, \text{ there exist } e', e'' \in C \cap D,\ e' < e'' \text{ such that } e' < e < e'' \text{ implies } e \notin C \cap D \text{ and } |\{e \in E : e' < e < e'' \text{ and } e \notin C \cup D\}| \text{ is even,} \tag{3.2}$$

then the alternating circuit signature $\mathcal{O}$ *of* $(E, \mathscr{C})$ *with respect to H is an orientation of* $(E, \mathscr{C})$.

Proof. Let $\mathcal{O}'$ be the cocircuit signature of M having for each $Y \in \mathcal{O}'$ with $\underline{Y} = \{e_{j_1}, \ldots, e_{j_t}\}$, $j_1 < j_2 < \cdots < j_t$, $sg_Y(e_{j_m}) = (-1)^{(m+j_m)-(l+j_l)} sg_Y(e_{j_l})$, for all $1 \leqslant l, m \leqslant t$. We will show that $\mathcal{O}, \mathcal{O}'$ satisfies (III), hence $\mathcal{O}$ is an orientation and $\mathcal{O}^\perp = \mathcal{O}'$. Let Y be as above and suppose that $X \in \mathcal{O}$ with $\underline{X} = \{e_{i_1}, \ldots, e_{i_s}\} \in \mathscr{C}$, $i_1 < i_2 < \cdots < i_s$, and $|\underline{X} \cap \underline{Y}| = 2$ or 3. Let e' and e'' satisfy (3.2) with $e' = e_{i_p} = e_{j_l}$ and $e'' = e_{i_q} = e_{j_m}$. Then

$$sg_X(e'') = (-1)^{q-p} sg_X(e')$$

and

$$sg_Y(e'') = (-1)^{(m+j_m)-(l+j_l)} sg_Y(e').$$

It suffices to show that $d = (q-p) + (m-l) + (j_m - j_l)$ is odd, since this implies that e' and e'' have opposite signs in one of X and Y and the same sign in the other. Let $S = \{e \in E : e' < e < e''\}$, so that $j_m - j_l = 1 + |S|$. Now $S \cap \underline{X} \cap \underline{Y} = \varnothing$ and $S \cap [E \backslash (\underline{X} \cup \underline{Y})] = c$ is even by (3.2). Therefore $j_m - j_l = 1 + c + |S \cap \underline{X}| + |S \cap \underline{Y}| = 1 + c + (q - p - 1) + (m - l - 1)$, so $d = 2(q-p) + 2(m-l) - 1 + c$.

The reader will note that $\mathcal{O}^\perp$ is obtainable from the alternating cocircuit signature of M with respect to H by reversing signs on either of the sets $\{e_h : h \text{ is odd}\}$ or $\{e_h : h \text{ is even}\}$.

EXAMPLE 3.8.1. Free matroids.
Let E be an n-element set, and suppose that $1 \leqslant r \leqslant n-1$. The *free matroid of rank* r on E, denoted $\mathscr{F}_n{}^r$, has as its bases all r-element subsets of E.

COROLLARY 3.9.1. *The free matroid* $\mathscr{F}_n{}^r$ *has, for each order* H *of its elements, an alternating orientation with respect to* H.

Proof. Let C and D be a circuit and cocircuit, respectively, of $\mathscr{F}_n{}^r$, so $|C| = r+1$ and $|D| = n - r + 1$. If $|C \cap D| = 2$, then $C \cup D = E$ and (3.2) is obviously satisfied for the pair C, D. Suppose that $C \cap D = \{x, y, z\}$, with $x < y < z$. Then $|E \backslash (C \cup D)| = 1$, say $\{e\} = E \backslash (C \cup D)$. If $e > y$, then $e' = x$ and $e'' = y$ satisfy (3.2), otherwise $e' = y$ and $e'' = z$, satisfy (3.2).

Corollary 3.9.1 can be established directly by verifying that for any n real numbers $t_1 < t_2 < \cdots t_n$, the matrix

$$\begin{bmatrix} t_1 & t_2 & \cdots & t_n \\ t_1{}^2 & t_2{}^2 & & t_n{}^2 \\ \vdots & \vdots & \cdots & \vdots \\ t_1^{r-1} & t_2^{r-1} & & t_n^{r-1} \end{bmatrix}$$

is an affine coordinatization of $\mathscr{F}_n{}^r$ that induces an alternating orientation.

An example of a nonfree matroid that satisfies the hypothesis of Proposition 3.9 is the matroid M determined by affine dependence in $\mathbb{R}^2$ on the six points in Fig. 1.

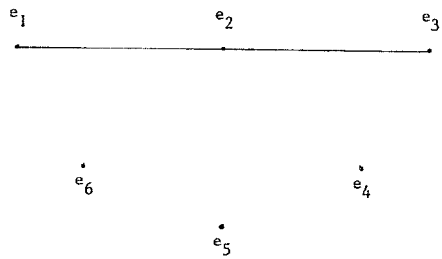

FIG. 1. A nonfree matroid with an alternating orientation.

EXAMPLE 3.10. The noncoordinatizable Vámos matroid is orientable.

Let $M_1 = (E, \mathcal{O}_1)$ be the oriented matroid determined by affine dependence over the reals on the set $E = \{e_1, ..., e_8\} \subseteq \mathbb{R}^3$ given in Fig. 2.

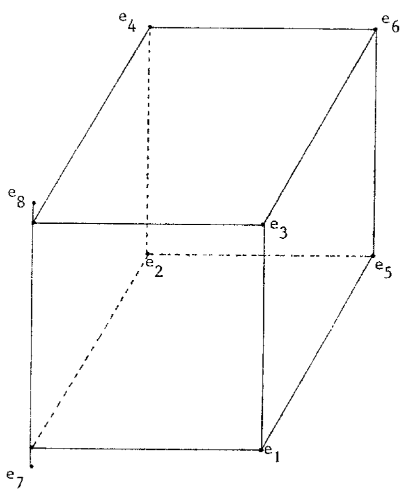

FIG. 2. A pre-Vámos oriented matroid M_1.

Note that six of the eight points in E are vertices of the unit cube, while e_7 and e_8 are translations of the remaining two vertices of the cube some small distance $\epsilon > 0$ along the line determined by that pair of vertices. Thus $\{e_3, e_4, e_6, e_8\}$ and $\{e_1, e_2, e_5, e_7\}$ are independent sets in $\underline{M}_1$. The circuit $C^* = \{e_1, e_2, e_3, e_4\}$ and the cocircuit $D^* = \{e_5, e_6, e_7, e_8\}$ of $\underline{M}_1$ play a special role in what follows.

Let $(E, \mathscr{C}_2)$ be the matroid having $\mathscr{C}_2 = (\underline{\mathcal{O}}_1 \cup \{C_5, C_6, C_7, C_8\}) \backslash \{C^*\}$, where $C_i = C^* + e_i$, $i = 5,\dots, 8$. Note that $\mathscr{C}_2{}^{\perp} = (\underline{\mathscr{O}}_1{}^{\perp} \cup \{D_1, D_2, D_3, D_4\})\backslash\{D^*\}$, where $D_j = D^* + e_j$, $j = 1,\dots, 4$. Vámos (see [9]) showed that $(E, \mathscr{C}_2)$ is not coordinatizable over any field (or division ring). We will exploit the close resemblance between $(E, \mathscr{C}_2)$ and the coordinatizable matroid $\underline{M}_1$ to describe how an orientation of $(E, \mathscr{C}_2)$ can be constructed. Let $\mathscr{O}_2$ and $\mathscr{O}_2{}'$ be a circuit signature and a cocircuit signature, respectively, of $(E, \mathscr{C}_2)$ having $X \in \mathscr{O}_2$ for all $X \in \mathscr{O}_1$ with $\underline{X} \in \mathscr{C}_2$ and $Y \in \mathscr{O}_2{}'$ for all $Y \in \mathscr{O}_1{}^{\perp}$ with $\underline{Y} \in \mathscr{C}_2{}^{\perp}$. The remaining circuits in $\mathscr{C}_2$ yet to be signed into $\mathscr{O}_2$ are C_5, C_6, C_7, C_8 and the remaining cocircuits are D_1, D_2, D_3, D_4. Let $X^* \in \mathscr{O}_1$ and $Y^* \in \mathscr{O}_1{}^{\perp}$ with $\underline{X}^* = C^*$ and $\underline{Y}^* = D^*$. For each C_i, $i = 5,\dots, 8$, include in $\mathscr{O}_2$ the signed sets X_i and $-X_i$ described by

$$\begin{aligned} sg_{X_i}(e) &= sg_{X^*}(e) && \text{if } e \neq e_i, \\ &= sg_{Y^*}(e) && \text{if } e = e_i, \end{aligned}$$

and for each Y_j, $j = 1,\dots, 4$ include in $\mathscr{O}_2{}'$ the signed sets Y_j and $-Y_j$ having

$$\begin{aligned} sg_{Y_j}(e) &= sg_{Y^*}(e) && \text{if } e \neq e_j, \\ &= -sg_{X^*}(e) && \text{if } e = e_j. \end{aligned}$$

Note that for $1 \leqslant j \leqslant 4$ and $5 \leqslant i \leqslant 8$, X_i and Y_j are orthogonal since e_i has the same sign in X_i and Y_j and e_j has opposite signs in X_i and Y_j. Moreover, for any $X \in \mathscr{O}_2$, $Y \in \mathscr{O}_2{}'$ such that either $\underline{X} \in \mathscr{C}_1$ or $\underline{Y} \in \mathscr{C}_1{}^{\perp}$, orthogonality of X and Y follows from the fact that M_1 is an oriented matroid. Therefore $\mathscr{O}_2$ is an orientation of the Vámos matroid $(E, \mathscr{C}_2)$ and $\mathscr{O}_2{}' = \mathscr{O}_2{}^{\perp}$.

The orientation $\mathscr{O}_2$ is given explicitly by the set of signed sets described in Table I and their opposites. Each entry in the table gives the index set of the elements in a circuit of the Vámos matroid. The signed set X represented by

TABLE I

The Orientation $\mathscr{O}_2$ of the Vámos Matroid

$1\overline{35}6$	$1\overline{23}45$	$123\overline{67}$	$\overline{12}567$	$\overline{1}3\overline{4}67$	$\overline{23}457$	$\overline{23}578$
$1\overline{37}8$	$1\overline{23}46$	$1\overline{23}68$	$12\overline{5}6\overline{8}$	$1\overline{34}68$	$234\overline{58}$	$\overline{23}678$
$2\overline{45}6$	$1\overline{23}4\overline{7}$	$\overline{12}457$	$\overline{12}578$	$145\overline{67}$	$\overline{23}467$	$345\overline{67}$
$2\overline{47}8$	$1\overline{23}4\overline{8}$	$1\overline{2}4\overline{58}$	$12\overline{67}8$	$14\overline{5}6\overline{8}$	$2\overline{34}68$	$\overline{34}568$
$5\overline{67}8$	$12\overline{357}$	$124\overline{67}$	$\overline{1}3\overline{4}57$	$\overline{14}578$	$235\overline{67}$	$34\overline{5}7\overline{8}$
	$1\overline{23}58$	$1\overline{2}4\overline{68}$	$134\overline{58}$	$\overline{14}678$	$23\overline{5}6\overline{8}$	$\overline{34}678$

an entry has as X^- those elements whose indices are overlined, e.g., $1\bar{3}\bar{5}6$ represents an oriented circuit X having $X^+ = \{1, 6\}$ and $X^- = \{3, 5\}$. The special circutis $C_5, \ldots, C_8$ of the Vámos matroid correspond to the first four entries in the second column of Table I. Table II similarly describes $\mathcal{O}_2^\perp$.

TABLE II

$\mathcal{O}_2^\perp$, the Dual of $\mathcal{O}_2$

1356	$\bar{1}56\bar{7}\bar{8}$	$12\bar{3}\bar{6}7$	$125\bar{6}7$	$13\bar{4}\bar{6}7$	$2\bar{3}457$	$2\bar{3}57\bar{8}$
1378	$256\bar{7}\bar{8}$	$1\bar{2}368$	$1256\bar{8}$	13468	$\bar{2}34\bar{5}8$	$2\bar{3}\bar{6}7\bar{8}$
2456	$356\bar{7}\bar{8}$	12457	$1257\bar{8}$	$\bar{1}456\bar{7}$	$2\bar{3}4\bar{6}7$	$3456\bar{7}$
2478	$\bar{4}56\bar{7}\bar{8}$	$\bar{1}24\bar{5}8$	$12\bar{6}78$	$1\bar{4}56\bar{8}$	23468	$34\bar{5}68$
$1\bar{2}3\bar{4}$	12357	$12\bar{4}\bar{6}7$	$13\bar{4}57$	$1\bar{4}57\bar{8}$	$\bar{2}356\bar{7}$	$34\bar{5}78$
	$1\bar{2}3\bar{5}8$	$\bar{1}2468$	$\bar{1}34\bar{5}8$	$\bar{1}46\bar{7}8$	$2\bar{3}56\bar{8}$	$346\bar{7}8$

The cocircuits $D_1, \ldots, D_4$ of the Vámos matroid correspond to the first four entries in the second column.

In demonstrating that $\mathcal{O}_2$ is an orientation we have relied on the structure of $\mathcal{O}_2$ and the fact that $\mathcal{O}_1$ is an orientation of M_1; we have not specifically invoked the sign properties of $\mathcal{O}_1$ that distinguish it from other orientations of M_1. Hence, any orientation of M_1 induces, as above, an orientation of M_2.

Other examples of noncoordinatizable orientable matroids are the non-Desargues matroid, [9, Example 2], and a modification of the non-Pappus matroid, [9, Example 3].

EXAMPLE 3.11. Some nonorientable matroids.

Let $r \geqslant 3$ be an integer and let E be a set of cardinality $2r$, $E = \{e_1, \ldots, e_r, e_1', \ldots, e_r'\}$. We denote by M_r the matroid on E with the following circuits: $\{e_i, e_i', e_j, e_j'\}$ for $1 \leqslant i < j \leqslant r$, $\{e_1, \ldots, e_{i-1}, e_i', e_{i+1}, \ldots, e_r\}$ for $1 \leqslant i \leqslant r$, $\{e_1', \ldots, e_r'\}$ and all $(r+1)$-subsets of E not containing any of the preceding $[r(r-1)/2] + r + 1$ sets.

LEMMA 3.11.1. *The involution τ on E defined by $\tau(e_i) = e_i'$ for $1 \leqslant i \leqslant r$ is an isomorphism from M_r to $(M_r)^\perp$.*

The proof is left to the reader.

Let $\mathbb{Q}$ denote the field of rational numbers.

LEMMA 3.11.2. *For all $e \in E$, $M_r \backslash e$ and M_r/e are coordinatizable over* $\mathbb{Q}$.

Proof. By the symmetry of M_r with respect to $e_1, \ldots, e_r$ and by Lemma 3.11.1 it suffices to prove Lemma 3.11.2 for $e = e_r'$.

A coordinatization of $M_r \backslash e_r'$. Let $\alpha_1, \ldots, \alpha_r$ be the canonical basis of $\mathbb{Q}^r$ and let $\alpha_i' = \alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r$ for $1 \leqslant i \leqslant r-1$. Then $M_r \backslash e_r'$ is isomorphic to the matroid on $\{\alpha_1, \ldots, \alpha_r, \alpha_1', \ldots, \alpha_{r-1}'\}$ determined by linear dependence in $\mathbb{Q}^r$.

A coordinatization of M_r/e_r'. Let $\alpha_1, \ldots, \alpha_{r-2}, \alpha_r$ be the canonical basis of $\mathbb{Q}^{r-1}$, $\alpha_{r-1} = \alpha_1 + \cdots + \alpha_{r-2}$, $\alpha_i' = \alpha_i + \alpha_r$ for $1 \leqslant i \leqslant r-2$, and $\alpha_{r-1}' = \alpha_{r-1} + (r-2)\alpha_r$. Then M_r/e_r' is isomorphic to the matroid on $\{\alpha_1, \ldots, \alpha_r, \alpha_1', \ldots, \alpha_{r-1}'\}$ determined by linear dependence in $\mathbb{Q}^{r-1}$.

PROPOSITION 3.12. *For $r \geqslant 4$, M_r is not orientable.*

Before proving Proposition 3.12 it will be useful to state the following simple consequence of the signed elimination property (II).

LEMMA 3.12.1. *Let $M = (E, \mathcal{O})$ be an oriented matroid. Suppose that $X_1, X_2 \in \mathcal{O}$, $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$, and that there is a unique circuit of $\underline{M}$ contained in $(\underline{X}_1 \cup \underline{X}_2) \backslash x$. Then there are exactly two signed circuits $X_3 \in \mathcal{O}$ and $-X_3 \in \mathcal{O}$ contained in $(\underline{X}_1 \cup \underline{X}_2) \backslash x$, and $(X_1^+ \cup X_2^+) \backslash (X_1^- \cup X_2^-) \subseteq X_3^+$, $(X_1^- \cup X_2^-) \backslash (X_1^+ \cup X_2^+) \subseteq X_3^-$.*

Proof of Proposition 3.12. Suppose, contrary to the proposition, that $\mathcal{O}$ is an orientation of M_r, $r \geqslant 4$. Denote by X_j, for $2 \leqslant j \leqslant r$, and Z signed circuits having $\underline{X}_j = \{e_1, e_1', e_j, e_j'\}$ and $\underline{Z} = \{e_1', e_2, \ldots, e_r\}$. By reversing signs in $\mathcal{O}$ on a subset of E and appropriately choosing each X_j and Z from the pair of opposite signed circuits having the given underlying circuit, we can assume with no loss of generality that $X_2^+ = \{e_1, e_1'\}$, $X_2^- = \{e_2, e_2'\}$, $e_1 \in X_j^+$ and $\{e_j, e_j'\} \subseteq X_j^-$ for $3 \leqslant j \leqslant r$, and $e_1' \in Z^-$.

Note that $C_j = \{e_2, e_2', e_j, e_j'\}$ is the unique circuit of M_r contained in $(\underline{X}_2 \cup \underline{X}_j) \backslash e_1$ (because $r \geqslant 4$). Since $e_1, e_1' \notin C_j$, $e_1 \in X_2^+ \cap (-X_j)^-$, and $e_1' \in X_2^+$, it follows from Lemma 3.12.1 that $e_1' \in (-X_j)^- = X_j^+$. Thus $X_j^+ = \{e_1, e_1'\}$, $X_j^- = \{e_j, e_j'\}$ for $j = 2, \ldots, r$.

Similarly, for $2 \leqslant j \leqslant r$ $\{e_1, \ldots, e_{j-1}, e_j', e_{j+1}, \ldots, e_r\}$ is the unique circuit of M_r contained in $(\underline{Z} \cup \underline{X}_j) \backslash e_1'$. By Lemma 3.12.1 $e_j \in Z^+$, hence $Z^- = \{e_1'\}$ and $Z^+ = \{e_2, \ldots, e_r\}$.

Now, by Lemma 3.11.1 $\{e_1, \ldots, e_r\}$ is a cocircuit of M_r. Let $Y \in \mathcal{O}^\perp$ have $\underline{Y} = \{e_1, \ldots, e_r\}$ and $e_1 \in Y^+$. It follows from orthogonality of $X_j \in \mathcal{O}$ and $Y \in \mathcal{O}^\perp$ that $e_j \in Y^+$, $j = 2, \ldots, r$, so $Y_j^- = \varnothing$. But this contradicts orthogonality of $Z \in \mathcal{O}$ and $Y \in \mathcal{O}^\perp$.

Proposition 3.12, Lemma 3.11.2, and Proposition 3.2 indicate that for all $r \geqslant 4$ M_r is not orientable, but all proper minors of M_r are orientable.

Therefore the matroids that collectively characterize orientable matroids by their exclusion as minors (in the spirit of [14]) are infinite in number. Examples of rank 3 nonorientable matroids with all proper minors orientable include the MacLane matroid (see [9]), as has been verified by Yves Kodratof (CNRS, Paris) with the aid of a computer, and the Fano matroid.

The matroids M_r are related to well-known matroids introduced by Lazarson (see [9]). Let $p \geqslant 2$ be a prime number, let $GF(p)$ be the Galois field $\mathbb{Z}/p\mathbb{Z}$, and let $E = \{e_1, ..., e_{p+1}, e_1', ..., e_{p+1}', f\} \subseteq (GF(p))^{p+1}$, where $\{e_1, ..., e_{p+1}\}$ is the canonical basis of $(GF(p))^{p+1}$, $f = e_1 + \cdots + e_{p+1}$, and $e_i' = f - e_i$, $i = 1, ..., p+1$. Let L_p be the matroid on E determined by linear dependence in $(GF(p))^{p+1}$. Lazarson showed that L_p is coordinatizable over a division ring F if and only if F has characteristic p. L_2 is the Fano matroid.

PROPOSITION 3.13. *For all prime numbers $p \geqslant 2$ M_{p+1} is isomorphic to $L_p \backslash f$.*

The proof is left to the reader.

It follows from Propositions 3.12 and 3.13 that for all prime numbers $p \geqslant 3$ the matroid L_p is not orientable. L_2 is also nonorientable as already noted.

Ingleton observed in [9] that for all prime numbers $p \geqslant 3$ $L_p \backslash f$ (and hence M_{p+1}) is coordinatizable over a division ring F if and only if F has characteristic p. It is not difficult to show that if the integer $p \geqslant 3$ is not prime, then M_{p+1} is not coordinatizable over any division ring.

4. MINORS OF ORIENTED MATROIDS

It is clear from Lemma 2.1.3 that minors of orientable matroids are orientable. In this section we will discuss oriented matroid minors. First we recall some notation from Section 2: (1) if X is a signed subset of E and $A \subseteq E$, then $(X \backslash A)$ denotes the signed set having $(X\backslash A)^+ = X^+ \backslash A$ and $(X \backslash A)^- = X^- \backslash A$; (2) if $\mathcal{O}$ is a collection of signed subsets of E, then $\mathrm{Min}(\mathcal{O})$ denotes the set of $X \in \mathcal{O}$ such that $\underline{X}$ is a (set-wise) minimal element of $\underline{\mathcal{O}}$.

PROPOSITION 4.1. *Let $M = (E, \mathcal{O})$ be an oriented matroid and let A and B be disjoint subsets of E. Then*

$$\hat{\mathcal{O}} = \mathrm{Min}\{X\backslash A : X \in \mathcal{O},\ X\backslash A \neq \varnothing \ \textit{and}\ \underline{X} \cap B = \varnothing\}$$

and

$$\hat{\mathcal{O}}^\perp = \mathrm{Min}\{Y\backslash B : Y \in \mathcal{O},\ Y\backslash B \neq \varnothing \ \textit{and}\ \underline{Y} \cap A = \varnothing\}$$

are matroid orientations and $\hat{\mathcal{O}}^\perp = (\hat{\mathcal{O}})^\perp$.

Proof. Note that $\underline{\hat{M}} = (E, \hat{\mathcal{O}})$ is the matroid minor of $\underline{M}$ obtained by contracting A and deleting B and $\underline{\hat{M}}^{\perp} = (E, \underline{\hat{\mathcal{O}}}^{\perp})$. It is easy to see that the pair $\hat{\mathcal{O}}$, $\hat{\mathcal{O}}^{\perp}$ satisfies the orthogonality property (IV), since any pair $\hat{X} \in \hat{\mathcal{O}}$, $\hat{Y} \in \hat{\mathcal{O}}^{\perp}$ corresponds to a pair $X \in \mathcal{O}$, $Y \in \mathcal{O}^{\perp}$ having $\hat{X} = X \backslash A$, $\hat{Y} = Y \backslash B$ and $\underline{X} \cap \underline{Y} \subseteq \underline{\hat{X}} \cap \underline{\hat{Y}}$. Thus it suffices to show that $\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}^{\perp}$ are signatures of $\hat{M}$. Clearly $\hat{\mathcal{O}} = -\hat{\mathcal{O}}$. We must show that $\hat{X}_1, \hat{X}_2 \in \hat{\mathcal{O}}$ and $\underline{\hat{X}_1} = \underline{\hat{X}_2}$ imply $\hat{X}_1 = \pm\hat{X}_2$. It then follows that $\hat{\mathcal{O}}$ is a circuit signature of $\underline{\hat{M}}$ and by symmetry $\hat{\mathcal{O}}^{\perp}$ is a cocircuit signature of $\underline{\hat{M}}$.

Suppose that $\hat{X}_1, \hat{X}_2 \in \hat{\mathcal{O}}$ and $\underline{\hat{X}_1} = \underline{\hat{X}_2}$. There exist $X_1, X_2 \in \mathcal{O}$ such that $\hat{X}_i = X_i \backslash A$ and $\underline{X}_i \cap B = \varnothing$ for $i = 1, 2$. Suppose that $\hat{X}_1 \neq \pm\hat{X}_2$, so there is an element $e \in (\hat{X}_1^+ \cap \hat{X}_2^-) \cup (\hat{X}_1^- \cap \hat{X}_2^+)$ and an element $e' \in (\hat{X}_1^+ \cap \hat{X}_2^+) \cup (\hat{X}_1^- \cap \hat{X}_2^-)$. Since $\mathcal{O}$ is an orientation and both e and e' have the same sign in X_i and $\hat{X}_i$, $i = 1, 2$, by the signed elimination property (II) for $\mathcal{O}$ there is some $X_3 \in \mathcal{O}$ having $X_3^+ \subseteq (X_1^+ \cup X_2^+) \backslash e$, $X_3^- \subseteq (X_1^- \cup X_2^-) \backslash e$, and $e' \in \underline{X}_3 \backslash A$. Therefore $\underline{X}_3 \subseteq (\underline{X}_1 \cup \underline{X}_2) \backslash e$, so $\underline{X}_3 \cap B = \varnothing$ and $e' \in (\underline{X}_3 \backslash A) \subseteq (\underline{X}_1 \cup \underline{X}_2) \backslash (A + e) = \underline{\hat{X}}_1 \backslash e$, contradicting the minimality of $\hat{X}_1$ in $\hat{\mathcal{O}}$.

Given $M = (E, \mathcal{O})$, A and B as in Proposition 4.1 we say that $\hat{M}$ is the *oriented matroid minor* of M obtained by *contracting* A and deleting B, and, of course, $\hat{M}^{\perp}$ is the oriented matroid minor of $M^{\perp}$ obtained by contracting B and deleting A. If $A = \varnothing$, then $\hat{\mathcal{O}} = \{X \in \mathcal{O} : \underline{X} \cap B = \varnothing\}$, denoted $\mathcal{O} \backslash B$, and if $B = \varnothing$, then $\hat{\mathcal{O}} = \operatorname{Min}\{X \backslash A : X \in \mathcal{O} \text{ and } \underline{X} \backslash A \neq \varnothing\}$, denoted $\hat{\mathcal{O}} = \mathcal{O}/A$. From the analogous properties of matroid minors we easily get

PROPOSITION 4.2. *Let M be an oriented matroid on a set E and let A and B be disjoint subsets of E. Then*

(i) $(M \backslash A) \backslash B = M \backslash (A \cup B)$;

(ii) $(M/A)/B = M/(A \cup B)$;

(iii) $(M/A) \backslash B = (M \backslash B)/A$.

Thus Proposition 4.1 can be restated in the form

THEOREM 4.3. *If $M = (E, \mathcal{O})$ is an oriented matroid and $A \subseteq E$, then M/A and $M \backslash A$ are oriented matroids and $(M/A)^{\perp} = M^{\perp} \backslash A$, $(M \backslash A)^{\perp} = M^{\perp}/A$.*

Proposition 4.2 and Lemma 2.1.2 imply the following useful result.

PROPOSITION 4.4. *If $M = (E, \mathcal{O})$ is an oriented matroid, $A \subseteq E$, $X \in \mathcal{O}$, and $x \in \underline{X} \backslash A$, then there exists $\hat{X} \in \mathcal{O}/A$ such that $x \in \underline{\hat{X}}$ and $\hat{X}^+ \subseteq X^+$, $\hat{X}^- \subseteq X^-$.*

5. CARRIERS AND SPANS OF ORIENTED MATROIDS

In this section we will discuss certain sets of signed sets whose minimal nonempty elements are the signed circuits of an oriented matroid. First we examine the effect of relaxing the requirement in (0) of Theorem 2.1 that $X_1, X_2 \in \mathcal{O}$ and $\underline{X}_2 \subseteq \underline{X}_1$ imply $X_1 = \pm X_2$.

PROPOSITION 5.1. *Let $\mathcal{O}$ be any set of nonempty signed sets such that $\mathcal{O}$ satisfies* (I) *and has $\mathcal{O} = -\mathcal{O}$.* Then for each *$X \in \mathcal{O}$ there exists $X' \in$* Min($\mathcal{O}$) *such that $X'^+ \subseteq X^+$ and $X'^- \subseteq X^-$.*

Proof. Let $X_1 \in \mathcal{O}$ have $X_1^+ \subseteq X^+$, $X_1^- \subseteq X^-$, and $|\underline{X}_1|$ as small as possible. If $X_1 \in \mathrm{Min}(\mathcal{O})$, then $X' = X_1$ satisfies the conclusion of the proposition. Suppose that $X_1 \notin \mathrm{Min}(\mathcal{O})$, so there exists $X_2 \in \mathcal{O}$ having $\underline{X}_2 \subsetneqq \underline{X}_1$. Select $X_2 \in \mathcal{O}$ such that $\underline{X}_2 \subsetneqq \underline{X}_1$ and $|(X_2^+ \cap X_1^-) \cup (X_2^- \cap X_1^+)|$ is minimized. Let $e \in (X_2^+ \cap X_1^-) \cup (X_2^- \cap X_1^+)$, which is nonempty by the choice of X_1. By (I) there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e$. But then $\underline{X}_3 \subsetneqq \underline{X}_1$ and $(X_3^+ \cap X_1^-) \cup (X_3^- \cap X_1^+) \subseteq [(X_2^+ \cap X_1^-) \cup (X_2^- \cap X_1^+)]\backslash e$, contradicting the choice of X_2.

THEOREM 5.2. *Let $\mathcal{O}$ be a set of nonempty signed sets such that $\mathcal{O}$ satisfies the elimination property* (I) *and has $\mathcal{O} = -\mathcal{O}$. Then* Min($\mathcal{O}$) *is the set of signed circuits of an oriented matroid.*

Proof. Clearly $X \in \mathrm{Min}(\mathcal{O})$ implies $X \neq \varnothing$ and $-X \in \mathrm{Min}(\mathcal{O})$.

Suppose that $X_1, X_2 \in \mathrm{Min}(\mathcal{O})$ with $\underline{X}_2 \subseteq \underline{X}_1$ and $X_1 \neq \pm X_2$. Let $e \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$, which is nonempty since $X_1 \neq \pm X_2$ and $\varnothing \neq \underline{X}_2 \subseteq \underline{X}_1$. By (I) there exists $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e$, so $\underline{X}_3 \subseteq \underline{X}_1\backslash e$, contradicting $X_1 \in \mathrm{Min}(\mathcal{O})$.

Now let $X_1, X_2 \in \mathrm{Min}(\mathcal{O})$, $X_1 \neq \pm X_2$, and $e \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$. By (I) there is some $X_3 \in \mathcal{O}$ such that $X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e$. By Proposition 5.1 there exists $X_3' \in \mathrm{Min}(\mathcal{O})$ such that $X_3'^+ \subseteq X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e$ and $X_3'^- \subseteq X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e$. Hence Min($\mathcal{O}$) satisfies (I).

Signed sets X_1, X_2 having $(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+) = \varnothing$ will be called *compatible*. The *union* $X_1 \cup X_2$ of compatible signed sets X_1 and X_2 is defined to be the signed set having $(X_1 \cup X_2)^+ = X_1^+ \cup X_2^+$ and $(X_1 \cup X_2)^- = X_1^- \cup X_2^-$. Given mutually compatible signed sets $X_1, \ldots, X_k$ in some set of signed sets $\mathscr{P}$, the union $X_1 \cup X_2 \cup \cdots \cup X_k$ is said to have a *conformal decompsition* in $\mathscr{P}$.

PROPOSITION 5.3. *If $\mathcal{O}$ is a set of signed sets such that $\mathcal{O}$ satisfies* ($\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}}$) *and has $\mathcal{O} = -\mathcal{O}$, then every $X \in \mathcal{O}$ has a conformal decomposition in* Min($\mathcal{O}$).

Proof. Suppose that $|X|$ is minimal subject to $X \in \mathcal{O}$ having no conformal decomposition in Min($\mathcal{O}$). There is no loss of generality in assuming $X^- = \varnothing$, since all of property ($\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}}$), Min($\mathcal{O}$), and the subset of $\mathcal{O}$ having conformal decompositions in Min($\mathcal{O}$) are invariant under the reversal of signs on any subset of E. By Proposition 5.1 there exists $X_1 \in \mathrm{Min}(\mathcal{O})$ having $X_1^+ \subseteq X^+$ and

$X_1^- = X^- = \varnothing$. It suffices to show that for each $e \in \underline{X}\backslash\underline{X}_1$ there is some $X_e \in \mathcal{O}$ having $X_e^- = \varnothing$ and $e \in X_e^+ \subsetneqq X^+$. Then by the choice of X each X_e has a conformal decomposition in $\mathrm{Min}(\mathcal{O})$ giving (with X_1) a conformal decomposition of X in $\mathrm{Min}(\mathcal{O})$.

Let $e \in \underline{X}\backslash\underline{X}_1$ and let $X_2 \in \mathcal{O}$ have $e \in X_2^+ \subsetneqq X^+ \cup X_1^- = X^+$, $X_2^- \subsetneqq X^- \cup X_1^+ = X_1^+$, and $|X_2^-|$ as small as possible (property $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$ ensures that we can find such an $X_2 \in \mathcal{O}$). Suppose $e' \in X_2^-$. Then by $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$ there exists $X_3 \in \mathcal{O}$ such that $e \in X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash e' \subseteq X^+$ and $X_3^- \subseteq (X_1^- \cup X_2^-)\backslash e' \subseteq X_2^-\backslash e'$, contradicting the choice of X_2. Thus $X_2^- = \varnothing$ and $e \in X_2^+ \subseteq X^+$.

THEOREM 5.4. *If $\mathcal{O}$ is a set of nonempty signed subsets of E that has $\mathcal{O} = -\mathcal{O}$, then $\mathcal{O}$ satisfies* $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$ *if and only if $\mathcal{O}$ satisfies* (II).

Proof. Clearly (II) implies $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$. Suppose that $\mathcal{O}$ satisfies $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$ and has $\mathcal{O} = -\mathcal{O}$. Let $X_1, X_2 \in \mathcal{O}$ with $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$ and $y \in (X_1^+ \cap X_2^+) \cup (X_1^- \cap X_2^-)$. We must show that there exists

$$X_3 \in \mathcal{O} \text{ such that } y \in \underline{X}_3,\ X_3^+ \subseteq (X_1^+ \cup X_2^+)\backslash x, \text{ and } X_3^- \subseteq (X_1^- \cup X_2^-)\backslash x. \tag{5.1}$$

By Proposition 5.3 X_1 and X_2 have conformal decompositions in $\mathrm{Min}(\mathcal{O})$; in particular there must exist $X_1', X_2' \in \mathrm{Min}(\mathcal{O})$ such that $X_i'^+ \subseteq X_i^+$, $X_i'^- \subseteq X_i^-$, and $y \in \underline{X}_i'$, $i = 1, 2$. If $x \notin \underline{X}_i'$ for $i = 1$ or 2, then $X_3 = X_i'$ satisfies (5.1). If $x \in \underline{X}_i' \cap \underline{X}_2'$, then it must be that $x \in (X_1'^+ \cap X_2'^-) \cup (X_1'^- \cap X_2'^+)$. Now, since $\mathcal{O} = -\mathcal{O}$ and $\mathcal{O}$ satisfies $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$, and hence (I), by Theorem 5.2 $\mathrm{Min}(\mathcal{O})$ is an oriented matroid. So by property (II) for $\mathrm{Min}(\mathcal{O})$, there exists $X_3 \in \mathrm{Min}(\mathcal{O})$ such that $y \in \underline{X}_3$, $X_3^+ \subseteq (X_1'^+ \cap X_2'^+)\backslash x$ and $X_3^- \subseteq (X_1'^- \cup X_2'^-)\backslash x$ and X_3 satisfies (5.1).

The reader will note that under the hypothesis of Theorem 5.4, the elimination property (I) is not equivalent to $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$ and (II). For example, let $E = \{1, 2, 3\}$ and let $\mathcal{O}$ consist of the six signed subsets of E described by

$$123,\ \overline{123},\ 2,\ \bar{2},\ 3,\ \bar{3}.$$

Then $\mathcal{O} = -\mathcal{O}$ and $\mathcal{O}$ satisfies (I), but not (II).

A set $\mathcal{O}$ of signed sets satisfying $(\mathrm{I}\frac{\mathrm{I}}{\mathrm{II}})$ and $\mathcal{O} = -\mathcal{O}$ will be called a *carrier* of the matroid orientation $\mathrm{Min}(\mathcal{O})$. Proposition 5.3 indicates that for a given oriented matroid $M = (E, \mathcal{O})$ every carrier of $\mathcal{O}$ is a subset of the set $\mathscr{K}(\mathcal{O})$, the *signed span of* $\mathcal{O}$, consisting of all signed subsets of E having conformal decompositions in $\mathcal{O}$.

If F is an ordered field and $\mathcal{O}$ is the set of signed supports of elementary vectors in a vector subspace $\mathscr{R} \subseteq F^E$, then $\mathscr{K}(\mathcal{O})$ is the set of signed supports of vectors in $\mathscr{R}$. In fact, for any oriented matroid $M = (E, \mathcal{O})$, $\mathscr{K}(\mathcal{O})$ retains many of the familiar properties that hold for the coordinatizable case. For example : $\varnothing \in \mathscr{K}(\mathscr{O})$, since the empty signed set decomposes into an empty union of signed circuits; $\mathscr{K}(\mathscr{O}) = -\mathscr{K}(\mathscr{O})$; $\mathscr{O} \subseteq \mathscr{K}(\mathscr{O})$; $\mathscr{K}(\mathscr{O})$ satisfies the elimination properties (I) and (II) of Theorem 2.1; and the pair $\mathscr{K}(\mathscr{O})$, $\mathscr{K}(\mathscr{O}^\perp)$ satisfies the painting property (V) and the orthogonality properties (III) and (IV) of Theorem 2.2 (in fact $\mathscr{K}(\mathscr{O})$ is precisely the set of all signed subsets X of E having X orthogonal to all $Y \in \mathscr{O}^\perp$). These conditions on signed spans of orientations are among the properties that Rockafellar [13] recognized oriented matroids ought to have.

The effect of contractions and deletions on the signed span of an orientation $\mathscr{O}$ on E is particularly easy to describe. Obviously $\mathscr{K}(\mathscr{O} \backslash e) = \{X \in \mathscr{K}(\mathscr{O}): e \notin \underline{X}\}$ for all $e \in E$. Furthermore, Proposition 4.4 implies that $X \backslash e \in \mathscr{K}(\mathscr{O}/e)$ for every $X \in \mathscr{O}$ and $e \in E$. Thus we have

PROPOSITION 5.5. *Let $M = (E, \mathscr{O})$ be an oriented matroid, let A and B be disjoint subsets of E, and let $\hat{\mathscr{O}} = (\mathscr{O}/A) \backslash B$. Then $\mathscr{K}(\hat{\mathscr{O}}) = \{X \backslash A: X \in \mathscr{K}(\mathscr{O})$ and $\underline{X} \cap B = \varnothing\}$.*

6. BINARY ORIENTED MATROIDS

Let E be a finite set. Recall that a subspace $\mathscr{R}$ of $\mathbb{R}^E$ is *unimodular* (*regular*) if all elementary vectors of $\mathscr{R}$ are proportional to $(0, \pm 1)$-vectors. A matroid M on E is *binary* if M is coordinatizable over $GF(2)$ and M is *unimodular* (*regular*) if $M = \underline{S}(\mathscr{R})$ for some unimodular subspace $\mathscr{R} \subseteq R^E$.

A *digraphoid* as defined by Minty in [12] is a dual pair of matroids M, $M^\perp$ together with circuit signatures $\mathscr{O}$ and $\mathscr{O}^\perp$ of M and $M^\perp$, respectively, such that the following axiom is satisfied:

$$\text{for all } X \in \mathscr{O},\ Y \in \mathscr{O}^\perp, \quad |(X^+ \cap Y^+) \cup (X^- \cap Y^-)| = |(X^+ \cap Y^-) \cup (X^- \cap Y^+)|.$$

It is obvious from the orthogonality property (IV) that a digraphoid is a dual pair of oriented matroids.

Actually digraphoids constituted the first attempt at axiomatizing oriented matroids. However, the above axiom is too restrictive—Minty showed that digraphoids are precisely the dual pairs of oriented matroids $S(\mathscr{R})$, $S(\mathscr{R}^\perp)$ for unimodular subspaces $\mathscr{R}$ of vector spaces $\mathbb{R}^E$ (see [12, App. 1]). The main result of this section is that binary oriented matroids are precisely the oriented matroids $S(\mathscr{R})$ that arise from unimodular subspaces $\mathscr{R}$ of $\mathbb{R}^E$, and digraphoids are, therefore, equivalent to dual pairs of binary oriented matroids.

THEOREM (Tutte [14, Proposition 7.51]). *A matroid M is unimodular if* *and only if M is binary and has no minor isomorphic to the Fano matroid (L_2 of Example 3.11) or its dual.*

PROPOSITION 6.1. *A binary matroid is orientable if and only if it is a unimodular matroid.*

Proof. Since minors of orientable matroids are orientable (see Section 4), an orientable matroid can have no minor isomorphic to the Fano, matroid or its dual. Hence by Tutte's theorem above a binary orientable matroid is unimodular. The converse is clear.

PROPOSITION 6.2. *Let M and M' be binary oriented matroids on a set E having $\underline{M} = \underline{M}'$. Then there exists $A \subseteq E$ such that $M' = {}_{\bar{A}}M$.*

In order to prove Proposition 6.2 we will first give some preliminary results.

Let M be a matroid on a set E. Whitney showed that the following two properties are equivalent [15, Theorem 19]:

(i) for all $x, y \in E$, $x \neq y$, there are circuits

$C_0, C_1, \ldots, C_k$ of M such that $x \in C_0$, $y \in C_k$ and $C_i \cap C_{i+1} \neq \varnothing$, for $i = 0, \ldots, k-1$;

and

(ii) for all $x, y \in E$, $x \neq y$, there is a circuit C of M containing x and y.

A matroid M having these properties is said to be *connected* (or *irreducible*).

A pair of circuits C, C' of a matroid M with rank function ρ is called *modular* if $\rho(C) + \rho(C') = \rho(C \cup C') + \rho(C \cap C')$.

LEMMA 6.2.1 (Tutte [14, Proposition 4.34]). *Let M be a matroid on a set E and let e be an element of E such that M/e is connected. Suppose that C and C' are distinct circuits of M having $e \in C \cap C'$. Then there are circuits $C = C_0$, $C_1, \ldots, C_k = C'$ of M such that $\{e\} \subsetneq C_i \cap C_{i+1}$ and the pair C_i, C_{i+1} is modular for $i = 0, 1, \ldots, k-1$.*

LEMMA 6.2.2. *Let M be a connected matroid on E with no 2-element circuits. Then there is an element $e \in E$ such that M/e is connected.*

Proof. The proof is by induction on $|E|$. The lemma is clearly true for $|E| = 3$. Suppose that $|E| \geqslant 4$. Crapo [6] showed that for every $e \in E$ either M/e or $M \backslash e$ is connected. Let $e \in E$ and suppose that M/e is not connected. Then $M \backslash e$ is connected, and by the inductive hypothesis there exists

$e' \in E\backslash e$ such that $(M\backslash e)/e'$ is connected. Since M is connected there is some circuit C of M having $e, e' \in C$. By the hypothesis of the lemma, $|C| \geqslant 3$, so $C\backslash e'$ is a circuit of M/e', $e \in C\backslash e'$, and $(C\backslash e') \cap (E\backslash\{e, e'\}) \neq \varnothing$. Since $(M\backslash e)/e' = (M/e')\backslash e$ is connected, it follows that M/e' is connected.

LEMMA 6.2.3 (Tutte [14, Proposition 5.35]). *A matroid is binary if and only if for all modular pairs of circuits C, C' of M such that $C \cap C' \neq \varnothing$ and $C \neq C'$, there are exactly three circuits contained in $C \cup C'$, namely, C, C', and $C \Delta C'$, the symmetric difference of C and C'.*

From Lemma 6.2.3 and the signed elimination property (I) we get

LEMMA 6.2.4. *Let $M = (E, \mathcal{O})$ be a binary oriented matroid. If $X, Z \in \mathcal{O}$, $\underline{X}, \underline{Z}$ is a modular pair in $\underline{M}$ and $x, z \in \underline{X} \cap \underline{Z}$, then $sg_X(x) \cdot sg_X(z) = sg_Z(x) \cdot sg_Z(z)$.*

We will need one more lemma. We say that a signed set X is *carried* by its underlying set $\underline{X}$.

LEMMA 6.2.5. *Let M and M' be binary oriented matroids on a set E having $\underline{M} = \underline{M}'$. Suppose that X_1, X_2 are distinct signed circuits of M such that $\underline{X}_1, \underline{X}_2$ is a modular pair of circuits and $e \in \underline{X}_1 \cap \underline{X}_2$.*

(i) *If X_1 and X_2 are signed circuits of M', then the opposite pair of signed circuits of M carried by $\underline{X}_1 \Delta \underline{X}_2$ are signed circuits of M'.*

(ii) *If $|\underline{X}_1 \cap \underline{X}_2| \geqslant 2$, X_1 is a signed circuit of M' and $X_2\backslash e$ is a signed circuit of M'/e, then X_2 is a signed circuit of M'.*

Proof. (i) By the signed elimination property (I), the signatures of the signed circuits of M and M' carried by $\underline{X}_1 \Delta \underline{X}_2$ are completely determined, in the same way, by X_1 and X_2, and are thus equal.

(ii) Let X_2' be a signed circuit of M' such that $\underline{X}_2 = \underline{X}_2'$ and $X_2\backslash e = X_2'\backslash e$ and let $x \in (\underline{X}_1 \cap \underline{X}_2)\backslash e$. By Lemma 6.2.4 we have $sg_{X_1}(e)\, sg_{X_1}(x) = sg_{X_2}(e)\, sg_{X_2}(x)$ and $sg_{X_1}(e)\, sg_{X_1}(x) = sg_{X_2'}(e)\, sg_{X_2'}(x)$. On the other hand $sg_{X_2}(x) = sg_{X_2'}(x)$, hence $sg_{X_2}(e) = sg_{X_2'}(e)$, and therefore $X_2 = X_2'$.

Proof of Proposition 6.2

The proof is by induction on $|E|$. Without loss of generality we may suppose that $|E| > 2$ and $\underline{M} = \underline{M}'$ is connected.

We consider two cases.

(1) Suppose first that $\underline{M} = \underline{M}'$ has a 2-element circuit $\{e, e'\}$. We have $\underline{M\backslash e} = \underline{M'\backslash e}$, hence by the inductive hypothesis there exists $A' \subseteq E\backslash e$ such that $M'\backslash e = {}_{\overline{A'}}(M\backslash e)$. Let X_0 and X_0' be signed circuits of M and M', respec-

tively, carried by $\{e, e'\}$ and having $e' \in X_0^+ \cap X_0'^+$. We set $A = A'$ if $X_0 = X_0'$ and $A = A' \cup \{e\}$ otherwise.

We show that $M' = {}_{\bar{A}}M$. Let X' be a signed circuit of M'. Since $M'\backslash e = {}_{\bar{A}}(M\backslash e)$ and $X_0' = {}_{\bar{A}}X_0$ we need only consider the case where $e \in \underline{X}'$ and $\underline{X}' \neq \{e, e'\}$. Now $\underline{X}_1 = \underline{X}'\Delta\{e, e'\} = \underline{X}'\backslash e + e'$ is a circuit of $\underline{M}\backslash e = \underline{M}'\backslash e$ and $\underline{X}_1$, $\{e, e'\}$ is a modular pair of circuits. Hence by (i) of Lemma 6.2.5, X' is a signed circuit of ${}_{\bar{A}}M$.

(2) Suppose now that $\underline{M} = \underline{M}'$ has no 2-element circuit. By Lemma 6.2.2 there exists $e \in E$ such that $\underline{M}/e = \underline{M}'/e$ is connected. Since $\underline{M/e} = \underline{M'/e}$, by the inductive hypothesis there exists $A' \subseteq E\backslash e$ such $M'/e = {}_{\bar{A'}}(M/e)$. Let X_0 be a signed circuit of M such that $e \in \underline{X}_0$. $|\underline{X}_0| \geqslant 2$, hence $X_0\backslash e$ is a signed circuit of M/e. Now ${}_{\bar{A'}}(X_0\backslash e)$ is a signed circuit of M'/e. Let X_0' be the signed circuit of M' such that $\underline{X}_0' = \underline{X}_0$ and $X_0'\backslash e = {}_{\bar{A'}}(X_0\backslash e)$. We set $A = A'$ if $X_0' = {}_{\bar{A'}}X_0$, $A = A' \cup \{e\}$ otherwise.

We will now show that $M' = {}_{\bar{A}}M$. Let X' be a signed circuit of M'. Since $X_0' = {}_{\bar{A}}X_0$ and $M'/e = {}_{\bar{A}}(M/e)$ we have only to consider the case where $X' \neq \pm X_0'$ and X' is not a circuit of M'/e.

(2a) $e \in \underline{X}'$.

By Lemma 6.2.1 there are signed circuits $X_1, X_2, \ldots, X_k = X$ of M such that $\underline{X}_k = \underline{X}'$, $\{e\} \subsetneqq \underline{X}_i \cap \underline{X}_{i+1}$, and $\underline{X}_i$, $\underline{X}_{i+1}$ is a modular pair of circuits, for $i = 0, 1, \ldots, k-1$. Now ${}_{\bar{A}}X_0 = X_0'$ is a signed circuit of ${}_{\bar{A}}M$ and M', ${}_{\bar{A}}X_1$ is a signed circuit of ${}_{\bar{A}}M$, and $({}_{\bar{A}}X_1)\backslash e$ is a signed circuit of M'/e, since $M'/e = {}_{\bar{A}}(M/e)$. Hence by (ii) of Lemma 6.2.5 ${}_{\bar{A}}X_1$ is a signed circuit of M'. By induction on k we show in this way that ${}_{\bar{A}}X_k = {}_{\bar{A}}X$ is a circuit of M'. Since $\underline{X} = \underline{X}'$, X' is a signed circuit of ${}_{\bar{A}}M$.

(2b) $e \notin \underline{X}'$.

There is a signed circuit X_1' of M' such that $e \in \underline{X}_1'$ and $\underline{X}_1'\backslash e \subsetneqq \underline{X}'$. $\underline{X}_1'$, $\underline{X}'$ is a modular pair of circuits. Since $\underline{M}'$ is binary there is a signed circuit X_2' of M' carried by $\underline{X}'\Delta\underline{X}_1'$ and we have $\underline{X}' = \underline{X}_1'\Delta\underline{X}_2'$. Now $e \in \underline{X}_1'$, $e \in \underline{X}_2'$, hence X_1' and X_2' are signed circuits of ${}_{\bar{A}}M$ by (2a). Therefore by (i) of Lemma 6.2.5 X' is a signed circuit of ${}_{\bar{A}}M$.

COROLLARY 6.2.6. *Let M be a binary oriented matroid on a set E. Then there is a unimodular subspace $\mathscr{R}$ of $\mathbb{R}^E$ such that $M = S(\mathscr{R})$.*

Proof. By Proposition 6.1 there is a unimodular subspace $\mathscr{R}$ of $\mathbb{R}^E$ such that $\underline{M} = \underline{S}(\mathscr{R})$. By Proposition 6.2 we have $M = {}_{\bar{A}}S(\mathscr{R})$ for some subset A of E. Hence $M = S(\epsilon\mathscr{R})$, where $\epsilon : E \to \{1, -1\}$ is defined by $\epsilon(x) = -1$ if $x \in A$ and $\epsilon(x) = 1$ if $x \in E\backslash A$.

From Proposition 6.2 and Corollary 6.2.6 we immediately get

COROLLARY 6.2.7 (Camion [3, Th. 4, Sect. 5.2], Brylawski and Lucas [2, Prop. 4.2]). *Let $\mathscr{R}$ and $\mathscr{R}'$ be unimodular subspaces of $\mathbb{R}^E$ having $\underline{S}(\mathscr{R}) = \underline{S}(\mathscr{R}')$. Then there is a mapping $\epsilon : E \to \{1, -1\}$ such that $\mathscr{R}' = \epsilon\mathscr{R}$.*

Corollary 6.2.7 is also implied by the recent work of both Bixby and Seymour on matroids coordinatizable over $GF(3)$.

COROLLARY 6.2.8. *Let G be an undirected graph. Then every orientation of the polygon–matroid (respectively, the bond-matroid) of G corresponds to some orientation of the edges of G.*

Minty's digraphoid axiom is the strengthening to the binary case of the orthogonality axiom (IV). It should be clear from Proposition 6.2 and its corollaries that the corresponding strengthening of the circuit elimination axiom (I) is:

> for all $X_1, X_2 \in \mathscr{O}$, $X_1 \neq -X_2$, having $(X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+) \neq \varnothing$, there exists $X_3 \in \mathscr{O}$ such that $X_3^+ \subseteq (X_1^+ \backslash X_2^-) \cup (X_2^+ \backslash X_1^-)$ and $X_3^- \subseteq (X_1^- \backslash X_2^+) \cup (X_2^- \backslash X_1^+)$.

Let M be an orientable matroid on a set E. The operation of sign reversal on subsets of E clearly describes an equivalence relation on the set of orientations of M. Proposition 6.2 indicates that if M is binary, then all pairs of orientations of M are related by sign reversal, i.e., there is exactly one class under this relation.

Problem. Let M be an orientable matroid. How many classes of orientations of M are there?

PROPOSITION 6.3. *Let n and r be positive integers, $2 \leqslant r \leqslant n-2$. Then the free matroid $\mathscr{F}_n{}^r$ of rank r on n elements has at least $(n-1)!/2$ classes of orientations.*

Proof. Let $\mathscr{O}$ be an orientation of a matroid M on a set E. Let $G(\mathscr{O})$ be the set of 2-element subsets $\{x, y\} \subseteq E$, $x \neq y$, such that either $sg_X(x) = sg_X(y)$ for all $X \in \mathscr{O}$ such that $\{x, y\} \subseteq \underline{X}$ or $sg_X(x) = -sg_X(y)$ for all $X \in \mathscr{O}$ such that $\{x, y\} \subseteq \underline{X}$. Clearly $G(\mathscr{O}) = G({}_{\bar{A}}\mathscr{O})$ for any $A \subseteq E$.

Let $M = \mathscr{F}_n{}^r$, $2 \leqslant r \leqslant n-2$, and let $\mathscr{O}$ be the alternating orientation of M with respect to some order $e_1 < e_2 < \cdots < e_n$ of E. It is easy to see that $G(\mathscr{O}) = \{\{e_i, e_{i+1}\} : i = 1, 2, \ldots, n, e_{n+1} = e_1\}$. Proposition 6.3 follows.

Note added in proof. Separate papers based on additional results from [1] ("A combinatorial abstraction of linear programming") and [10] ("Bases of oriented matroids" and "Convexity in oriented matroids") will appear in this journal.

Note added in proof. Recently, previously unpublished work on oriented matroids by the late Jon Folkman has appeared in summary from in the Ph.D. Thesis of Jim Lawrence (University of Washington, Seattle, Summer 1975). Although Folkman's approach to oriented matroids differs noticeably from ours, his axiomatization is based on an elimination property that is clearly equivalent to (II) of our Theorem 2.1. Thus it is clear that the axiomatizations represented by Theorems 2.1 and 2.2, each of which one or both of us had developed before learning of Folkman's work, are equivalent to Folkman's axiomatization. It appears from his unpublished notes that Folkman was aware of the possibility of an axiomatization of oriented matroids based on the orthogonality property (IV) of Theorem 2.2, but, apparently, he never pursued it.

Acknowledgment

This paper is a synthesis of work that was undertaken separately by the authors and was, in each case, completed in early 1974, shortly before we learned from S. B. Maurer of our common interests. Our initial announcements of these results appeared in [1, 10].

References

1. R. G. Bland, "Complementary Orthogonal Subspaces of $\mathbb{R}^n$ and Orientability of Matroids," Ph.D. Dissertation, Cornell University, May 1974.
2. T. H. Brylawski and D. Lucas, "Uniquely Representable Combinatorial Geometries," preprint, 1973.
3. P. Camion, "Matrices totalement unimodulaires et problèmes combinatoires," Thèse, Université de Bruxelles, 1963.
4. P. Camion, Modules unimodulaires, *J. Combinatorial Theory* **4** (1968), 301–362.
5. A. Cheung and H. H. Crapo, A combinatorial perspective on algebraic geometry, *Advances in Math.* **20** (1976), 388–414.
6. H. H. Crapo, A higher invariant for matroids, *J. Combinatorial Theory* **2** (1967), 406–417.
7. H. H. Crapo and G.-C. Rota, "Combinatorial Geometries," M.I.T. Press, Cambridge, Mass., 1970.
8. D. R. Fulkerson, Networks, frames, blocking systems, *in* "Mathematics of the Decision Sciences" (G. B. Dantzig and A. F. Veinott, Jr., Eds.), pp. 303–334, Amer. Math. Soc., Providence, R.I., 1968.
9. A. W. Ingleton, Representation of matroids, *in* "Combinatorial Mathematics and Its Applications" (D. J. A. Welsh, Ed.), pp. 149–167, Academic Press, London/New York, 1971.
10. M. Las Vergnas, "Matroïdes orientables," preprint, April 1974, announced in *C. R. Acad. Sci. Paris* **280** (1975), 61–64.
11. A. Lehman, A solution to the Shannon switching game, *J. SIAM* **12** (1964), 687–725.
12. G. J. Minty, On the axiomatic foundations of the theories of directed linear graphs, electrical networks, and network programming, *J. Math. Mech.* **15** (1966), 485–520, reprinted in "Graph Theory," M. A. A. Study No. 12, (D. R. Fulkerson, Ed.), Math. Assoc. of America, 1975.
13. R. T. Rockafellar, The elementary vectors of a subspace of R^n, *in* "Combinatorial Mathematics and Its Applications," Proc. of the Chapel Hill Conf. (1967) (R. C. Bose and T. A. Dowling, Eds.), pp. 104–127, Univ. of North Carolina Press, Chapel Hill, 1969.
14. W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Stand. B* **69** (1965), 1–47.
15. H. Whitney, On the abstract properties of linear dependence, *Amer. J. Math.* **57** (1935), 507–553.

Printed in Belgium